Price-taking Equilibrium and Convergence in Games

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ABSTRACT

Similarities between Walrasian equilibrium in quasilinear models of exchange and correlated equilibrium in normal form games are demonstrated via their common conjugate duality properties. Similarities extend to *tâtonnement*-like methods of convergence in games. Important contrasts related to decentralization are exhibited between *tâtonnement* and utility maximizing price-taking formulations of fictitious play as algorithms for finding the minimum of a convex optimization problem that characterizes convergence to correlated equilibrium.

Keywords:
Economics: quasi-linear utility, Walrasian equilibrium, *tâtonnement*.

Games: normal form, correlated equilibrium, Nash equilibrium, fictitious play, regret.

Convex Analysis: conjugate duality, Fenchel's Duality Theorem, subgradient algorithm.

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1 INTRODUCTION

The well-known equivalence between the Minimax Theorem for two-person zero-sum games and linear programming marks the connection between game theory and modern duality (Gale, Kuhn, Tucker [1951]). More recently, correlated equilibrium (Aumann [1974]) has been characterized as a minimax (Hart and Schmeidler [1989]) or linear programming problem (Nau and McCardle [1990] and Myerson [1997]). An earlier expression of duality is the relation between prices and quantities in the Walrasian model of general equilibrium. When utilities are quasilinear, price-taking, i.e., Walrasian, equilibria can be characterized as the primal and dual solutions to a linear programming problem (having a finite number of constraints and, when commodities are divisible, an infinite number of activities) or a minimax problem. (See, below). These observations frame the objective of this paper: to formulate a game from the perspective of the (quasilinear) general equilibrium model of exchange and to show that correlated equilibria can be regarded as price-taking.

More specifically, probabilistic play of a game are *prices*, deviations are *trades* and the satisfaction of incentive compatibility constraints is the *market-clearing condition* for price-taking equilibrium in games. The bridge between demand functions in economics and best response functions in games is built on their conjugate duality properties employed in convex analysis. Correlated equilibrium is demonstrated as an application of Fenchel's Duality Theorem (Fenchel [1951], Rockafellar [1970]). As added support for this objective, connections between *tâtonnement*, a method of convergence to Walrasian equilibrium, and fictitious play methods of convergence to correlated equilibrium in games (Foster and Vohra [1997], Fudenberg and Levine [1998], Hart and Mas-Colell [2000]) are demonstrated.

The overall goal is to formalize common features of price-taking in economics and games that will also serve to delineate their differences. For example, the property of prices in an exchange economy that they describe a condition in which aggregate excess demands are zero can be said to describe a *pre-condition* for equilibrium. The convention is that the desired exchanges will take place once the pre-condition is met, after which no one wants to trade. In a game, there is no pre-condition — equilibrium prices are such that no individual wants to trade/deviate. By itself, this difference is not dispositive: equilibrium in an exchange economy can be readily redefined to satisfy the stronger condition that no one wants to trade. (See Remark 2, below.) Nevertheless, the different descriptions of equilibrium are a point of departure as to how disequilibrium is modeled and convergence is achieved.

Convergence in an exchange economy focuses on prices adjusting to the infeasibility of aggregate excess demands until they satisfy the equilibrium pre-condition of feasibility. In a game, all plays and deviations from them are feasible and convergence necessarily leads from one feasible outcome to another. To accommodate the different approaches to feasibility two (price-quantity) dualities will be used to analyze games, called (E) for 'exchange' and (F) for 'fictitious play.' The E-duality for games is designed to mimic the price quantity duality in economics: everyone faces the same prices, trades in the same 'commodity' space, and price adjustments focus on the *sum* of excess demands, as in the economic model of exchange. This is in contrast to the F-duality where price adjustments are based on the *product* of excess demands, as in the game-theoretic algorithm of fictitious play. The dualities are formally equivalent: either can be used to characterize and demonstrate the existence of correlated equilibrium and either imply, with suitable modifications, convergence to correlated equilibrium. However, these two dualities for games exhibit contrasting properties when compared to *tâtonnement* in exchange.

Tâtonnement in economic models expresses the informational economy of the price system; namely, individuals need only know the current prices of commodities and their own utilities to make their utility maximizing choices. Moreover, price changes are commodity-specific in the sense that adjustments to the price of any commodity depends only on that commodity's excess demands. But in the E-duality for games, excess demands are defined by the *utility consequences* of one's deviations; hence, excess demand reducing price adjustments are based on utilities. Therefore, they do not exhibit the same informational economy as *tâtonnement* in economics. In addition, prices changes for one n-tuple of individual deviations/trades may depend on excess demands in all the others.

The F-duality for games follows *tâtonnement* in economics: it is trades/deviations themselves, rather than their utility consequences, which play the role of excess demands. When combined with fictitious play, however, it does more. The informational economy of *tâtonnement* in economics is subject to an obvious qualification. Since they are price-takers, individuals cannot change prices. This calls for another entity having knowledge of aggregate excess demands, often referred to as an auctioneer, to implement price changes. Consequently, *tâtonnement* in economics is not completely decentralized (or, in the language of computer algorithms, not completely distributed).

This contrasts with fictitious play (and the F-duality) for games in which, since prices are based on previous choices, *price changes emerge autonomously as the updating of those observations.* In other words, fictitious play offers a more decentralized description of price adjustment in games than does *tâtonnement* in economics. (This more complete decentralization imposes considerable observational and record-keeping burdens on individuals that could, under certain conditions, be more efficiently discharged by a central authority responsible for collecting and disseminating that information.)

Fictitious play can include histories of simultaneous play consistent with Aumann's extension of Nash's non-cooperative equilibrium. The need for further modification is based on the well-known fact that best responses in a game cannot be assumed to be continuous functions of prices/probabilities. This problem is avoided in economics because standard treatments of *tâtonnement* assume excess demands vary continuously with prices. With differentiability and quasi-linearity, *tâtonnement* corresponds to a gradient method for finding equilibrium as the minimum of a convex function. As illustrated, below, discontinuities in price-taking utility maximizing demands — resulting, for example, from indivisible commodities — cause problems for convergence in economic models of exchange. They can be cured by modifying *tâtonnement* to keep track of time and the history of previous trades, i.e., to look more like fictitious play.

Further modification of fictitious play in games relies on various methods of smoothing to eliminate discontinuities. The formulation adopted, below, uses a canonical quadratic cost method for converting non-differentiable into differentiable optimization problems. In the F-duality, price-taking utility maximization reproduces Hart and Mas-Colell's 'regret-matching.' Further, by extending a finite player game to a population with a continuum of individuals of each type — which is especially appropriate for the price-taking hypothesis that single individuals would have no influence on prices, a somewhat stronger version of population convergence to correlated equilibrium is demonstrated. Hence, *tâtonnement*-like conclusions in E-duality, where price changes require an Auctioneer's active participation (see the need for a projection mapping in Section 5.3), are replaced by price changes in the F-duality emerging directly from individuals' desires to exploit perceived profit opportunities.

The following section introduces the conjugate duality framework for price-taking equilibrium in a quasilinear exchange economy. This is the model for characterization and existence of correlated equilibrium, using the E-duality, in Section 3. Differences in efficiency properties between economics and games are characterized and discussed in Section 4. Section 5 is devoted to *tâtonnement* convergence, both in economics and games. Second 6 introduces the F-duality, autonomous models of price adjustment provided by randomized fictitious play, the formulation of regret-matching as utility maximization and its convergence properties in both the finite and population models of a game. This is followed by a Concluding Remark. A polyhedral version of Fenchel's Duality Theorem is stated in the Appendix.

2 Price-taking Equilibrium in Exchange

The purpose of this section is to highlight features of price-taking equilibrium with quasilinearity as a template for games.

2.1 Model

The characteristics of individual i are defined by the utility function for (non-money) commodities,

$$v_i : \mathbb{R}^\ell \to \mathbb{R} \cup \{-\infty\}. \tag{2.1.1}$$

The characteristics include the set of trades i can feasibly make, defined by

$$Z_i := \{z_i : v_i(z_i) > -\infty\} \subset \mathbb{R}^\ell. \tag{2.1.2}$$

Denote by $\mathbf{0}$ the zero element of $\mathbb{R}^\ell$. Assume throughout that

- $\mathbf{0} \in Z_i$ and $v_i(\mathbf{0}) = 0$
- Z_i is compact
- v_i is continuous on Z_i.

In addition to trades z_i in (non-money) commodities, there is a "money" commodity, $m_i \in \mathbb{R}$. The utility of (z_i, m_i) is of the quasilinear form

$$v_i(z_i) + m_i. \tag{2.1.3}$$

Since the money commodity enters each individual's utility in the same way, the data of quasilinear exchange economy can be summarized as

$$\mathscr{E} = \langle v_i \rangle := (v_1, v_2, \ldots, v_n). \tag{2.1.4}$$

The smallest concave function greater than or equal to v_i is

$$\widehat{v}_i(z_i) := \sup\Big\{\sum_k \lambda_k v_i(z_i^k) : \sum_k \lambda_k z_i^k = z_i, \lambda_k \geq 0, \sum_k \lambda_k = 1\Big\}. \tag{2.1.5}$$

Hence,

$$\widehat{Z}_i = \Big\{\sum_k \lambda_k z_i^k : z_i^k \in Z_i, \lambda_k \geq 0, \sum_k \lambda_k = 1\Big\} \tag{2.1.6}$$

is the set of trading possibilities for $\widehat{v}_i$, and v_i is concave when $v_i = \widehat{v}_i$.

REMARK 1: (ADDED ASSUMPTIONS ON Z_i) Standard assumptions for $\mathscr{E}$ are

- $Z_i = \widehat{Z}_i$ (commodities are divisible)
- $\mathbf{0} \in \operatorname{int} Z_i$ (individuals can trade all commodities).

When Z_i is finite, commodities are indivisible, implying that $\widehat{v}_i$ is polyhedral, i.e., concavification derived from a finite set. Polyhedral convexity will be a property of games, below, but polyhedral concavity is not a standard assumption in exchange. A relevant exception is the asssignment model. In that case, a supplier can offer one unit of a commodity personalized to that seller and each buyer can purchase at most one unit from any seller. In comparison to standard assumptions, above, which imply the relative interior with respect to the smallest affine set common to all trades is

$$\operatorname{rint}\Big(\bigcap_i Z_i\Big) = \mathbb{R}^\ell,$$

personalization of commodities in the assignment model implies

$$\operatorname{rint}\Big(\bigcap_i Z_i\Big) = \{\mathbf{0}\}.$$

Personalization of deviations/trades will also be a property of games. (Optimal solutions to the assignment model are known to follow from the Minimax Theorem (von Neumann [1953]).)

2.2 PRICE-TAKING UTILITY MAXIMIZATION IN $\mathscr{E}$

At prices $p \in \mathbb{R}^\ell$ for non-money commodities and a normalized price of 1 for the money commodity, the individual's budget constraint is $p \cdot z_i + m_i = 0$. The maximum utility achievable at p, or indirect utility, is

$$\begin{aligned}\sup_{(z_i,m_i)} \{v_i(z_i) + m_i : p \cdot z_i + m_i = 0\} &= \sup_{z_i} \{v_i(z_i) - p \cdot z_i\} \\ &= -\inf_{z_i} \{p \cdot z_i - v_i(z_i)\} \\ &:= -v_i^*(p)\end{aligned} \tag{2.2.1}$$

The expression $v_i^*(p) = \inf_{z_i} \{p \cdot z_i - v_i(z_i)\}$ is the concave conjugate of v_i. Therefore, $-v_i^*(p)$ is convex.

The demand correspondence (the argmax of $-v_i^*(p)$) is:

$$\partial v_i^*(p) = \{z_i : (q-p)\cdot z_i \geq v_i^*(q) - v_i^*(p),\ \forall q\} \tag{2.2.2}$$

When v_i^* is differentiable, $\partial v_i^*(p) = \nabla v_i^*(p)$. With quasilinear utility the marginal utility of money income $\equiv 1$. Hence, (2.2.2) is *Roy's Identity* for quasilinear utility without the requirement that utility is differentiable.

The inverse demand correspondence for $z_i \in \text{dom}\ v_i$, i.e., the prices (if any) at which it would be utility maximizing to choose z_i, is

$$\partial v_i(z_i) = \{p : p\cdot(y_i - z_i) \geq v_i(y_i) - v_i(z_i),\ \forall y_i\}. \tag{2.2.3}$$

Three descriptions of price-taking utility maximization are:

FACT 1: The following are equivalent:

- $-v_i^*(\bar{p}) = v_i(\bar{z}_i) - \bar{p}\cdot\bar{z}_i$
- $\bar{p} \in \partial v_i(\bar{z}_i)$
- $\bar{z}_i \in \partial v_i^*(\bar{p})$

Price-taking maximization implies that if $\widehat{v}_i(\bar{z}_i) > v_i(\bar{z}_i)$, there is no p at which an individual with utility v_i would choose $\bar{z}_i$. But maximizing choices of v_i are similar to $\widehat{v}_i$ in the sense that if $\partial v_i(\bar{z}_i) \neq \emptyset$, then $v_i(\bar{z}_i) = \widehat{v}_i(\bar{z}_i)$ since $\bar{p} \in \partial v_i(\bar{z}_i)$ implies

$$v_i^*(\bar{p}) = \bar{p}\cdot\bar{z}_i - v_i(\bar{z}_i) = \inf_{z_i}\{\bar{p}\cdot z_i - \widehat{v}_i(z_i)\}. \tag{2.2.4}$$

2.3 Conjugate Duality Characterization of Price-taking Equilibrium in $\mathscr{E}$

The maximum gains to $\mathscr{E}$ facing the aggregate resource constraint z is

$$V_{\mathscr{E}}(z) = \sup\Big\{\sum_i v_i(z_i) : \sum_i z_i = z\Big\}. \tag{2.3.1}$$

Unlike $v_i(\mathbf{0}) = 0$, $V_{\mathscr{E}}(\mathbf{0}) \geq 0$. The inequality is strict when there are positive gains from trade among individuals. The analog of the normalization $v_i(\mathbf{0}) = 0$ for $\mathscr{E}$ is

$$\mathscr{V}_{\mathscr{E}}(z) := V_{\mathscr{E}}(z) - V_{\mathscr{E}}(\mathbf{0}). \tag{2.3.2}$$

The conjugate of $\mathscr{V}_{\mathscr{E}}$ is

$$\mathscr{V}^*_{\mathscr{E}}(p) = \inf_p \left\{ p \cdot z - \mathscr{V}_{\mathscr{E}}(z) \right\} = -\sup_p \left\{ \mathscr{V}_{\mathscr{E}}(z) - p \cdot z \right\}. \tag{2.3.3}$$

Like $-v^*_i(p)$, $-\mathscr{V}^*_{\mathscr{E}}(p)$ is convex.

Ignoring the normalization that the price of the money commodity is 1, there is no restriction on $p \in \mathbb{R}^\ell$. For purposes of symmetry, below, the absence of such restriction is formalized by the concave indicator of $\mathbb{R}^\ell$,

$$\mathscr{P}(p) = \begin{cases} 0 & \text{if } p \in \mathbb{R}^\ell, \\ -\infty & \text{otherwise.} \end{cases} \tag{2.3.4}$$

Its conjugate is

$$\mathscr{P}^*(z) = \inf_p \left\{ p \cdot z - \mathscr{P}(p) \right\}, \tag{2.3.5}$$

Hence, $\mathscr{P}^*(z) > -\infty$ if and only if $z = \mathbf{0}$.

Denote a list of individual trades as $\langle z_i \rangle = (z_1, z_2, \ldots, z_n)$. For a quasilinear model of exchange, the Walrasian definition of equilibrium as utility maximization and market clearance is:

DEFINITION 1: $(p^0, \langle z^0_i \rangle)$ is a *price-taking (Walrasian) equilibrium* for $\mathscr{E} = \langle v_i \rangle$ if

- $v_i(z^0_i) - p^0 \cdot z^0_i = \sup_{(z_i, m_i)} \left\{ v_i(z_i) + m_i : p^0 \cdot z_i + m_i = 0 \right\}, \quad \forall i$
- $\sum_i z^0_i = \mathbf{0}$

PROPOSITION 1: (CHARACTERIZATION)
$(p^0, \langle z^0_i \rangle)$ is a price-taking equilibrium for $\mathscr{E}$ if and only if $\sum_i z^0_i = \mathbf{0}$ and

$$\min_p \left\{ -\mathscr{V}^*_{\mathscr{E}}(p) - \mathscr{P}(p) \right\} = -\mathscr{V}^*_{\mathscr{E}}(p^0) - \mathscr{P}(p^0) = -\mathscr{V}^*_{\mathscr{E}}(p^0) = 0.$$

Equivalently, $\sum_i v^*_i(p^0) = \sum_i v_i(z^0_i)$.

Note the role of prices in *minimizing* the gains from trade. Since the opportunity to trade at any prices implies $-\mathscr{V}^*_{\mathscr{E}}(p) \geq 0$, the minimum is achieved by prices encouraging $\mathscr{E}$ not to trade.

Proof. (From saddle-point to price-taking equilibrium) Let $p^0 \in \partial \mathcal{V}_{\mathscr{E}}(\mathbf{0})$. Then

$$V_{\mathscr{E}}(z) - V_{\mathscr{E}}(\mathbf{0}) \le p^0 \cdot z.$$

Let $\sum_i z_i^0 = \mathbf{0}$, $\sum_i v_i(z_i^0) = V_{\mathscr{E}}(\mathbf{0})$. Therefore, if $\sum_i z_i = z$, then $\sum_i v_i(z_i) \le V_{\mathscr{E}}(z)$, and

$$\sum_i v_i(z_i) - v_i(z_i^0) \le p^0 \cdot \sum_i (z_i - z_i^0) = p^0 \cdot z.$$

Hence, the inequality

$$v_i(z_i) - v_i(z_i^0) \le p^0 \cdot (z_i - z_i^0),$$

must hold for all i and z_i. Therefore, z_i^0 is utility maximizing at p^0.

(From price-taking equilibrium to saddle-point) Reverse the argument. Price-taking maximization at p^0 implies

$$v_i(z_i) - v_i(z_i^0) \le p^0 \cdot (z_i - z_i^0),$$

for all i and z_i, from which it is readily concluded that

$$\sum_i v_i(z_i^0) = V_{\mathscr{E}}(\mathbf{0}),$$

and $p^0 \cdot z \ge V_{\mathscr{E}}(z) - V_{\mathscr{E}}(\mathbf{0})$ for all $z = \sum_i z_i$, i.e., $p^0 \in \partial \mathcal{V}_{\mathscr{E}}(\mathbf{0})$. □

Equilibrium does not exist if

(1) $\widehat{V_{\mathscr{E}}}(\mathbf{0}) > V_{\mathscr{E}}(\mathbf{0})$, i.e., $V_{\mathscr{E}}$ is 'not effectively concave' at $\mathbf{0}$; or,

(2) $\widehat{V_{\mathscr{E}}}(\mathbf{0}) = V_{\mathscr{E}}(\mathbf{0})$, but $\partial \widehat{V_{\mathscr{E}}}(\mathbf{0}) = \emptyset$, i.e., the marginal gain from trade at $\mathbf{0}$ is not bounded above.

(1) is the substantive qualification, while (2) is a more technical condition that is precluded if $\mathbf{0} \in \operatorname{int} Z$. An illustration of (2) is:

Example 1 : A single individual economy with $Z_i = Z = [0,1]$ and $v_i(z) = v(z) = 2z^{1/2}$, $z \in Z$. (Here $\mathbf{0} = 0$.) The directional derivative of v at $\mathbf{0}$ is $-\infty$ in the negative direction and $dv(z)/dz = (z_i)^{-1/2}$, $0 < z < 1$. Hence, the directional derivative of v at $\mathbf{0}$ in the positive direction is ∞. Even though v is concave and continuous on Z, $\mathbf{0}$ is on the boundary of Z.

From Proposition 1, existence depends entirely on $\partial \mathcal{V}_{\mathscr{E}}(\mathbf{0}) \neq \emptyset$. By construction, $\partial \mathcal{V}_{\mathscr{E}}(\mathbf{0}) = \partial V_{\mathscr{E}}(\mathbf{0})$. Under standard assumptions (see Remark 1, above), the practical conditions to achieve this requirement is $v_i = \widehat{v}_i$ to satisfy (1), along with a Lipschitz condition on v_i to satisfy (2).

REMARK 2: (INDIVIDUAL MINIMAX DESCRIPTION OF EQUILIBRIUM) The definition of price-taking equilibrium highlights prices as guides for individuals about which trades to make, while equilibrium prices stipulate the mutual compatibility of their plans. Equilibrium in games, below, focuses on each individual's desire not to deviate from an as-if no-trade position. Price-taking in $\mathscr{E}$ can be similarly reformulated.

Let

$$\nu_i(y_i \mid z_i) := \nu_i(z_i + y_i) - \nu_i(z_i), \tag{2.3.6}$$

be the change in utility to i by making the 'deviation' y_i from $z_i \in Z_i$. Thus, $\nu_i(y_i \mid \mathbf{0}) = \nu_i(y_i)$. Price-taking utility maximization from z_i is

$$-\nu_i^*(p \mid z_i) = \sup_{y_i} \{\nu_i(y_i \mid z_i) - p \cdot y_i\} = -\inf_{y_i} \{p \cdot y_i - \nu_i(y_i \mid z_i)\} \tag{2.3.7}$$

At z_i, $(\bar{p}, \bar{y}_i)$ is a saddle-point of $\nu_i(y_i \mid z_i) - p \cdot y_i$ for i when

$$-\nu_i^*(p \mid z_i) \geq -\nu_i^*(\bar{p} \mid z_i) = \nu_i(\bar{y}_i \mid z_i) - \bar{p} \cdot \bar{y}_i \geq \nu_i(y_i \mid z_i) - \bar{p} \cdot y_i, \quad \forall p, \forall y_i. \tag{2.3.8}$$

The following individual, rather than aggregate, saddle-point condition replaces emphasis on the mutual compatibility of demands in Proposition 1, where i's initial position is $\mathbf{0}$, with an equivalent 'stay-there' requirement, i.e., $y_i = \mathbf{0}$, with respect to i's finally attained position, z_i^0, that more closely resembles the description of correlated equilibrium in games, below.

DEFINITION 2: $(p^0, \langle z_i^0 \rangle)$ is a *price-taking equilibrium* for $\mathscr{E}$ if

- $-\nu_i^*(p \mid z_i^0) \geq -\nu_i^*(p^0 \mid z_i^0) = \nu_i(\mathbf{0} \mid z_i^0) - p^0 \cdot \mathbf{0} = 0 \geq \nu_i(y_i \mid z_i^0) - p^0 \cdot y_i, \quad \forall p, \forall y_i, \forall i.$
- $\sum_i z_i^0 = \mathbf{0}$.

REMARK 3: (SELECTION PROBLEM) Price-taking equilibrium is often interpreted as implying that knowledge of prices suffices for individuals to achieve equilibrium. This view is supported when prices determine utility maximizing choices *uniquely*, i.e., when ν_i^* is differentiable at p^0 and therefore $\partial \nu_i^*(p^0) = \{z_i^0\}$. However, when $\partial \nu_i^*(p^0)$ is not a singleton, e.g., when $\widehat{\nu}_i$ is 'flat' at z_i^0 and $\partial \nu_i^*(p^0)$ is a convex set in Z_i, the choice in $\partial \nu_i^*(p^0)$ is critical. (With production and constant returns to scale, this well-known problem is particularly acute.) With respect to equilibrium, the issue is the difference between 'is' and 'can be.' The definition of price-taking equilibrium ignores this problem by identifying ‘can be’ as ‘is,’ i.e., precluding choices $z_i \in \partial v_i^*(p^0)$, $i = 1,\dots,n$, at equilibrium prices which do not sum to $\mathbf{0}$. Nevertheless, the *selection problem*—which utility maximizing choice to make—is the source of continuity issues with consequences for convergence to price-taking equilibrium in games. (See Example 2, below.)

3 Price-taking Equilibrium in Games

Analysis of games (in normal form) is organized to parallel exchange.

3.1 Model

Let A_i denote the finite set of actions a_i, $i = 1,\dots,n$ and

$$A := A_1 \times \cdots \times A_n. \tag{3.1.1}$$

As in $\mathbb{R}^\ell$, the zero element of $\mathbb{R}^A$ is $\mathbf{0}$. The same notation to denote prices in an exchange economy, where $p \in \mathbb{R}^\ell$, is used to denote prices in a game where $p \in \mathbb{R}^A$. The normalized set of such prices are the probabilities

$$P := \Big\{p : A \to \mathbb{R}_+ : \sum_a p(a) = 1\Big\}. \tag{3.1.2}$$

Similar notation will be used to define utility/payoff functions in a game. The difference is that unlike the non-linear function $v_i : \mathbb{R}^\ell \to \mathbb{R} \cup \{-\infty\}$, the utility of i is defined by $\mathfrak{v}_i : A \to \mathbb{R}$, an element of $\mathbb{R}^A$. The payoff to i from p is written as

$$p \cdot \mathfrak{v}_i = \sum_a p(a)\mathfrak{v}_i(a). \tag{3.1.3}$$

A game $\mathscr{G}$ in *normal form* is a pair $(\langle \mathfrak{v}_i \rangle, A)$, or simply

$$\mathscr{G} := \langle \mathfrak{v}_i \rangle, \tag{3.1.4}$$

taking A as given.

3.2 PRICE-TAKING UTILITY MAXIMIZATION IN $\mathscr{G}$

This section and the following describe a reformulation of a game in the language of exchange.

3.2.1 *Deviations as Trades and the E-duality*

Let

$$\mathbf{D}_i = \{\mathbf{d}_i : A_i \to A_i\}, \tag{3.2.1}$$

be the set of mappings from A_i to itself. Included in $\mathbf{D}_i$ is the identity $\mathbf{d}_i^{\mathrm{Id}}$, i.e., $\mathbf{d}_i^{\mathrm{Id}}(a_i) = a_i, \forall a_i$. Use $\mathbf{D}_i$ to define

$$D_i := \{d_i : d_i(a) = \mathfrak{v}_i(\mathbf{d}_i(a_i), a_{-i}) - \mathfrak{v}_i(a_i, a_{-i}), \mathbf{d}_i \in \mathbf{D}_i\}. \tag{3.2.2}$$

The set D_i represents the utility consequences to i when deviating from A_i and choosing according to $\mathbf{d}_i \in \mathbf{D}_i$ instead. An alternative interpretation is that the analog of trades Z_i in $\mathscr{E}$ is D_i in $\mathscr{G}$, where $d_i = \mathbf{0}$ is the choice not to trade/deviate.

As in the assignment model version of $\mathscr{E}$ where $\operatorname{rel\,int}\left(\bigcap_i Z_i\right) = \{\mathbf{0}\}$ (see Remark 1), since each i is the only one controlling D_i,

$$\operatorname{rint}\Big(\bigcap_i D_i\Big) = \{\mathbf{0}\}.$$

The indicator function of D_i is

$$\mathbf{v}_i(d_i) = \begin{cases} 0 & \text{if } d_i \in D_i, \\ \infty & \text{if } d_i \notin D_i. \end{cases} \tag{3.2.3}$$

Note that $\mathbf{v}_i(\mathbf{0}) = 0$, i.e., not deviating, is always feasible.

With trading opportunities given by $\mathbf{v}_i$, the maximum utility i can achieve at prices p (the indirect utility/conjugate function) is

$$\mathbf{v}_i^*(p) := \sup_{d_i} \left\{p \cdot d_i - \mathbf{v}_i(d_i)\right\} = \sup\left\{p \cdot d_i : d_i \in D_i\right\}, \tag{3.2.4}$$

where

$$p \cdot d_i = \sum_a p(a) d_i(a). \tag{3.2.5}$$

Discreteness of D_i implies that $\mathbf{v}_i$ is not convex. Nevertheless, as the sup of linear functions, $\mathbf{v}_i^*$ is convex.

The demand correspondence (the argmax of $\mathbf{v}_i^*(p)$) is given by the subdifferential of $\mathbf{v}_i^*(p)$ as

$$\partial \mathbf{v}_i^*(p) := \{\bar{d}_i : (q-p)\cdot \bar{d}_i \le \mathbf{v}_i^*(q) - \mathbf{v}_i^*(p),\ \forall q\}. \tag{3.2.6}$$

The inverse demand correspondence for $\bar{d}_i \in D_i$ are the prices (if any) at which it is utility maximizing to choose $\bar{d}_i$,

$$\partial \mathbf{v}_i(\bar{d}_i) := \{p : p\cdot(d_i - \bar{d}_i) \le \mathbf{v}_i(d_i) - \mathbf{v}_i(\bar{d}_i),\ \forall d_i \in \mathbb{R}^A\}. \tag{3.2.7}$$

In parallel with Fact 1 for $\mathscr{E}$, the conditions (i) $\mathbf{v}_i^*(\bar{p}) = \bar{p}\cdot\bar{d}_i - \mathbf{v}_i(\bar{d}_i)$, (ii) $\bar{p} \in \partial \mathbf{v}_i(\bar{d}_i)$, and (iii) $\bar{d}_i \in \partial \mathbf{v}_i^*(\bar{p})$ are equivalent.

The E-duality for games is the pairing (p, d_i). Comparing $(p, d_i) \in \mathbb{R}^A \times \mathbb{R}^A$ for $\mathscr{G}$ with the $(p, z_i) \in \mathbb{R}^\ell \times \mathbb{R}^\ell$ duality for $\mathscr{E}$, since $\mathbf{0} \in D_i \subset \mathbb{R}^A$ and $\mathbf{0} \in Z_i \subset \mathbb{R}^\ell$,

$$\mathbf{v}_i^*(p) \ge 0,\ \forall p \in \mathbb{R}^A \quad \text{and} \qquad -\nu_i^*(p) \ge 0,\ \forall p \in \mathbb{R}^\ell.$$

For the purpose of characterizing correlated equilibrium via the E-duality, it will suffice to describe $\mathscr{G} = \langle \mathfrak{v}_i \rangle$ by $\langle \mathbf{v}_i \rangle$, the indicator functions of $\langle D_i \rangle$.

REMARK 4: (ELIMINATION OF MONEY TRANSFERS IN $\mathscr{G}$) The conjugate function $\nu_i^*(p) = \inf_{z_i}\{p\cdot z_i - \nu_i(z_i)\}$ for $\mathscr{E}$ includes the utility function $\nu_i(z_i)$ and the budget constraint defined by prices $p \in \mathbb{R}^\ell$ and money transfers $m_i = -p\cdot z_i$. For the conjugate $\mathbf{v}_i^*(p) = \sup_{d_i}\{p\cdot d_i - \mathbf{v}_i(d_i)\}$ in $\mathscr{G}$, the indicator function $\mathbf{v}_i$ is effectively the 'budget constraint' limiting i's choices, while $p\cdot d_i$ is the valuation of d_i at prices $p \in \mathbb{R}^A$, i.e., there are no money transfers in $\mathscr{G}$.

REMARK 5: (REVEALED PREFERENCE) In $\mathscr{E}$, information from utility maximizing choices (p, z_i), where $z_i \in \partial \nu_i^*(p)$, can be used to derive the utility function ν_i underlying those choices (assuming $\nu_i = \widehat{\nu}_i$). This revealed preference conclusion does not hold for $\mathscr{G}$. Although D_i is determined by $\mathfrak{v}_i$, the latter cannot be recovered from the former. Thus, if $\tilde{\mathfrak{v}}_i(a) = \alpha_i \mathfrak{v}_i(a) + f_i(a_{-i})$, $\alpha_i > 0$, it is readily established that if $\mathbf{v}_i^*$ is the conjugate of $\mathbf{v}_i$ derived from $\mathfrak{v}_i$ and $\tilde{\mathbf{v}}_i^*$ is the conjugate of $\tilde{\mathbf{v}}_i$ derived from $\tilde{\mathfrak{v}}_i$, then although $\tilde{\mathfrak{v}}_i \ne \mathfrak{v}_i$, $\partial \tilde{\mathbf{v}}_i^*(p) = \partial \mathbf{v}_i^*(p)$, i.e., the same choices are made by individuals with non-equivalent utilities.

3.3 E-Duality Characterization of Price-taking Equilibrium in $\mathscr{G}$

The analog for $\mathscr{G}$ of $Z = \sum_i Z_i$ for $\mathscr{E}$ is the indicator of $D = \sum_i D_i$,

$$\begin{aligned}\mathbf{V}_{\mathscr{G}}(d) &:= \begin{cases} 0 & \text{if } d \in D, \\ \infty & \text{if } d \notin D, \end{cases} \\ &= \inf\Big\{\sum_i \mathbf{v}_i(d_i) : \sum_i d_i = d\Big\}. \end{aligned} \tag{3.3.1}$$

Its conjugate is

$$\mathbf{V}^*_{\mathscr{G}}(p) := \sup_d \{p \cdot d - \mathbf{V}_{\mathscr{G}}(d)\} = \sum_i \mathbf{v}^*_i(p). \tag{3.3.2}$$

For exchange, the prices of non-money commodity are given by the (unrestricted) indicator function $\mathscr{P}(p)$. For games, P is the relative price normalization of non-negative, non-zero prices in $\mathbb{R}^A$. The concave indicator function of P is

$$\mathbf{P}(p) := \begin{cases} 0 & \text{if } p \in P, \\ -\infty & \text{if } p \notin P. \end{cases} \tag{3.3.3}$$

The concave conjugate of $\mathbf{P}$ is the (concave) support function of P,

$$\mathbf{P}^*(d) = \inf_p \{p \cdot d - \mathbf{P}(p)\} = \inf\{p \cdot d : p \in P\}. \tag{3.3.4}$$

Definition 3: A *correlated equilibrium* for $\mathscr{G}$ is a $p^0 \in P$ such that no individual would gain by deviating, i.e.,

$$p^0 \cdot D_i \leq 0, \quad \forall i.$$

In price-taking terminology, the absence of gain in correlated equilibrium is equivalently described as

$$\mathbf{0} \in \partial \mathbf{v}^*_i(p^0) \Longleftrightarrow p^0 \in \partial \mathbf{v}_i(\mathbf{0}), \quad \forall i. \tag{3.3.5}$$

This observation immediately implies:

PROPOSITION 2: *(*CHARACTERIZATION*)*
$(p^0, \langle d_i^0 \rangle)$, where $d_i^0 = \mathbf{0}$, is a correlated equilibrium if and only if

$$\mathbf{V}^*_{\mathscr{G}}(p^0) - \mathbf{P}(p^0) = \inf_p \left\{\mathbf{V}^*_{\mathscr{G}}(p) - \mathbf{P}(p)\right\} = 0.$$

Equivalently, $\mathbf{V}^*_{\mathscr{G}}(p^0) = \sum_i \mathbf{v}^*_i(p^0) = 0.$

Concavity and Lipschitz qualifications are required for existence of price-taking equilibrium in $\mathscr{E}$. Added qualification is not required in $\mathscr{G}$.

PROPOSITION 3: *(*EXISTENCE*)*

$$P^0 = \operatorname{argmin}\left\{\mathbf{V}^*_{\mathscr{G}}(p) - \mathbf{P}(p)\right\}$$

is non-empty (and convex).

Proof. Fenchel's Duality Theorem (see the Appendix), a conjugate duality extension of the Minimax Theorem, is used to demonstrate existence.

(I) Extend $\mathbf{V}_{\mathscr{G}}$ to the indicator of the smallest convex cone containing $D = \sum_i D_i$, i.e.,

$$\vec{\mathbf{V}}_{\mathscr{G}}(d) = \begin{cases} 0 & \text{if } d = \sum_i \sum_k \lambda_i^k d_i^k, d_i^k \in D_i, \lambda_i^k \geq 0, \\ \infty & \text{otherwise.} \end{cases} \tag{3.3.6}$$

Therefore, if $p \in P$,

$$\vec{\mathbf{V}}^*_{\mathscr{G}}(p) = \sup_d \left\{p \cdot d - \vec{\mathbf{V}}_{\mathscr{G}}(d)\right\} < \infty \Longrightarrow \vec{\mathbf{V}}^*_{\mathscr{G}}(p) = 0 = p \cdot \mathbf{0}. \tag{3.3.7}$$

The expression

$$\vec{\mathbf{V}}^*_{\mathscr{G}}(p) - \mathbf{P}(p) \tag{3.3.8}$$

is the difference between the polyhedral convex function $\vec{\mathbf{V}}^*_{\mathscr{G}}(p)$ and the polyhedral concave function $\mathbf{P}(p)$. And

$$\mathbf{P}^*(d) - \vec{\mathbf{V}}_{\mathscr{G}}(d) \tag{3.3.9}$$

is the difference between the concave function $\mathbf{P}^*$, the conjugate of $\mathbf{P}$, and the convex function $\vec{\mathbf{V}}_{\mathscr{G}}$, which is the conjugate of $\vec{\mathbf{V}}^*_{\mathscr{G}}$. The latter conclusion follows from the result that the conjugate of $\vec{\mathbf{V}}^*_{\mathscr{G}}$, called the biconjugate, equals $\vec{\mathbf{V}}_{\mathscr{G}}$ when $\vec{\mathbf{V}}_{\mathscr{G}}$ is the indicator function of a polyhedral convex set (Rockafellar [1970, Theorem 14.1]. As the indicator function of a non-null set that includes D, $\vec{\mathbf{V}}_{\mathscr{G}}(d) = 0$ is a proper convex function, i.e., taking values $> -\infty$. Its conjugate $\vec{\mathbf{V}}^*_{\mathscr{G}}$ is also proper since $\vec{\mathbf{V}}^*_{\mathscr{G}}(\mathbf{0}) = 0$. And

$$\mathbf{P}^*(\mathbf{0}) = 0 = \vec{\mathbf{V}}_{\mathscr{G}}(\mathbf{0}); \tag{3.3.10}$$

i.e., $\mathbf{0}$ *belongs to the effective domains of* $\mathbf{P}^*$ *and* $\vec{\mathbf{V}}_{\mathscr{G}}$. Therefore, by Fenchel's Duality Theorem, there exists p^0 and d^0 such that

$$\min_p \left\{\vec{\mathbf{V}}^*_{\mathscr{G}}(p) - \mathbf{P}(p)\right\} = \vec{\mathbf{V}}^*_{\mathscr{G}}(p^0) - \mathbf{P}(p^0) = \mathbf{P}^*(d^0) - \vec{\mathbf{V}}_{\mathscr{G}}(d^0) = \max_d \left\{\mathbf{P}^*(d) - \vec{\mathbf{V}}_{\mathscr{G}}(d)\right\}. \tag{3.3.11}$$

The equalities (3.3.7) and (3.3.11) imply the minimax equality

$$\vec{\mathbf{V}}^*_{\mathscr{G}}(p^0) - \mathbf{P}(p^0) = \mathbf{V}^*_{\mathscr{G}}(p^0) = p^0 \cdot d^0 = 0 = \mathbf{P}^*(d^0) - \vec{\mathbf{V}}_{\mathscr{G}}(d^0) = \mathbf{P}^*(d^0). \tag{3.3.12}$$

Therefore, $p^0 \cdot d^0 = 0$ and it can be achieved by setting $d^0 = \mathbf{0}$. But this can also be achieved at $\mathbf{V}_{\mathscr{G}}(\mathbf{0}) = 0$, i.e.,

$$\mathbf{V}^*_{\mathscr{G}}(p^0) - \mathbf{P}(p^0) = 0 = \mathbf{P}^*(\mathbf{0}) - \mathbf{V}_{\mathscr{G}}(\mathbf{0}). \tag{3.3.13}$$

(III) Convexity is a well-known and readily established property of argmin$\left\{\mathbf{V}^*_{\mathscr{G}}(p) - \mathbf{P}(p)\right\}$. □

REMARK 6: (CONVEXITY OF P) The critical condition for the existence of price-taking equilibrium in $\mathscr{E}$ is $\partial V_{\mathscr{E}}(\mathbf{0}) \neq \emptyset$. Similarly, existence in $\mathscr{G}$ requires $\partial \mathbf{V}_{\mathscr{G}}(\mathbf{0}) \neq \emptyset$. The difference is that unless $V_{\mathscr{E}} = \widehat{V}_{\mathscr{E}}$, the requirement cannot typically be fulfilled for $\mathscr{E}$. *Convexity of P eliminates the need for convexifying* $\mathbf{V}_{\mathscr{G}}$, i.e., $\mathbf{V}_{\mathscr{G}}$ serves as an effective substitute for $\widehat{\mathbf{V}}_{\mathscr{E}}$.

REMARK 7: (NASH AND CORRELATED EQUILIBRIA) Let

$$Q = \left\{p(a_1, \dots, a_n) = q_1(a_1) \times \cdots \times q_n(a_n) : q_i(a_i) \geq 0, \sum_{a_i} q_i(a_i) = 1, \forall i\right\} \subset P$$

be the set of individually independent probability mixtures. Nash showed, in effect, that for all $\mathscr{G}$,

$$P^0 \cap Q \neq \emptyset.$$

I.e., there exists $p^0 = (q_1^0, q_2^0, \dots, q_n^0) \in Q$ such that $\mathbf{0} \in \partial \mathbf{v}_i^*(p^0)$, $\forall i$. Hence, if $P^0 = \{p^0\}$, p^0 must be a Nash equilibrium. Moreover, in 2-person zero-sum games,

$$P^0 \cap P \text{ is utility equivalent to } P^0 \cap Q.$$

These properties make Nash equilibrium an n-person extension of non-cooperative equilibrium for 2-person zero-sum games. Nevertheless, the latter also includes games for which $P^0 \backslash Q \neq \emptyset$. To illustrate with matching pennies, add the null choice (N) for the column player yielding a zero payoff to each. Let $p^0 = (1/8, 1/8, 1/8, 1/8, 1/3, 1/6) \in P$, where the first four elements correspond to the original matching pennies choices and the last two are for (H,N) and (T,N). It is readily confirmed that p^0 is a correlated equilibrium, but $p^0 \notin Q$.

	H	T	N
H	1, −1	−1,1	0,0
T	−1,1	1, −1	0,0

Adding $P^0 \backslash Q$ to $P^0 \cap Q$ in 2-person zero-sum games and extending to n-person (non-zero) sum games yields the minimax characterization of correlated equilibrium as the generalization of equilibrium for 2-person zero-sum games.

4 Properties of Equilibrium in $\mathscr{E}$ and $\mathscr{G}$

Formal similarities between price-taking equilibria in $\mathscr{E}$ and $\mathscr{G}$ belie well-known significant differences.

4.1 Multiplicity and Efficiency of Equilibria in $\mathscr{E}$

Denote the set of price-taking equilibria for $\mathscr{E}$ as $P^0 \times \{\mathbf{0}\} \subset \mathbb{R}^\ell \times Z$. P^0 determines the set of equilibrium payoffs

$$\mathscr{E}[P^0] = \left\{ \langle -v_i^*(p) \rangle : p \in P^0 \right\} \subset \mathbb{R}^n. \tag{4.1.1}$$

Relevant points of comparison of this set with games are:

Proposition 4: *(Utility Consequences of Equilibria in $\mathscr{E}$)*

(a) $\mathscr{E}[P^0]$ *is a convex set*

$$(b) \qquad \mathscr{E}[P^0] \subset \Big\{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \sum_i \alpha_i = V_{\mathscr{E}}(\mathbf{0})\Big\} \subset \mathbb{R}^n.$$

Convexity of $\mathscr{E}[P^0]$ follows from the well-known convexity of P^0 and the convexity of $-\nu_i^*$. The hyperplane property (b) follows from the fact that for every price-taking equilibrium p^0,

$$-\sum_i \nu_i^*(p^0) = V_{\mathscr{E}}(\mathbf{0}).$$

Thus, $\mathscr{E}$ exhibits a (maximum) constant-sum property with respect to price-taking equilibrium payoffs, but the distribution of the sum is unique only if P^0 is a singleton.

4.2 Multiplicity and Inefficiency of Equilibria for $\mathscr{G}$

The payoff consequences of equilibria in $\mathscr{G}$ are

$$\mathscr{G}[P^0] = \Big\{\langle p^0 \cdot \mathfrak{v}_i \rangle : p^0 \in P^0\Big\} \subset \mathbb{R}^n. \tag{4.2.1}$$

Note that expected gain to i from p^0, $p^0 \cdot \mathfrak{v}_i$, is typically *not* the same as the maximum deviation gain to i from p^0, $\mathbf{v}_i^*(p^0)$.

In the quasilinear model $\mathscr{E}$, efficient allocations can be defined by maximizing a weighted sum of utilities where the weights are fixed and equal to 1. In $\mathscr{G}$, the weights may vary. Let $\langle \lambda_i \rangle$, $\lambda_i > 0$ for all i, and define

$$\overline{\mathscr{G}}_{\langle \lambda_i \rangle} := \max \Big\{ p \cdot \sum_i \lambda_i \mathfrak{v}_i : p \in P \Big\}, \tag{4.2.2}$$

as the maximum gains in $\mathscr{G}$ when $\langle \mathfrak{v}_i \rangle$ are weighted by $\langle \lambda_i \rangle$.

In comparison to Proposition 5 for $\mathscr{E}[P^0]$, well-known properties of $\mathscr{G}[P^0]$ are:

Proposition 5: *(Utility Consequences of Equilibria in $\mathscr{G}$)*

$$(a) \qquad \mathscr{G}[P^0] \textit{ is a convex set}$$

Typically,

$$(b) \qquad \mathscr{G}[P^0] \textit{ is not contained in a hyperplane}$$

$$(c) \qquad p \in P, \quad p \cdot \sum_i \lambda_i \mathfrak{v}_i = \overline{\mathscr{G}}_{\langle \lambda_i \rangle} \Longrightarrow p \notin P^0$$

Property (a) is shared with $\mathscr{E}[P^0]$; but not (b) and (c).

REMARK 8: (PRICE-TAKING AND EXTERNALITIES) The economic rationale for inefficiency in $\mathscr{G}$ is attributable to utility functions $\mathfrak{v}_i(a_i,a_{-i})$ allowing j's choice a_j, $j \neq i$, to impose 'externalities' on i. Similar consequences would follow if the utility functions $\nu_i(z_i)$ in $\mathscr{E}$ were modified to $\mathscr{E}_{\text{ext}} = \langle \overset{e}{\nu}_i(z_i; z_{-i}) \rangle$. There is, however, a difference between the way externalities are modeled in $\mathscr{G}$ and $\mathscr{E}_{\text{ext}}$.

In $\mathscr{G}$, prices $p(a) = p(a_i,a_{-i})$ exist for all $a \in A$. But the built-in restriction in a normal form game is that 'property rights' for i are such that deviations/trades are *always* restricted to $\mathbf{d}_i : A_i \to A_i$. Hence, the indirect utility/conjugate function is

$$\mathbf{v}_i^*(p) = \sup_{d_i} \left\{ p \cdot d_i - \mathbf{v}_i(d_i) \right\} = \sup_{\mathbf{d}_i \in \mathbf{D}_i} \left\{ p \cdot [\mathfrak{v}_i(\mathbf{d}_i) - \mathfrak{v}_i] \right\},$$

The consequences of externalities in $\mathscr{G}$ are exhibited by comparing the gains to i from $\mathbf{d}_i$, $p \cdot d_i = p \cdot [\mathfrak{v}_i(\mathbf{d}_i) - \mathfrak{v}_i]$, with the gains from $\mathbf{d}$, where

$$\mathfrak{v}_i(\mathbf{d})(a_1,a_2,\ldots,a_n) := \mathfrak{v}_i(\mathbf{d}_1(a_1),\mathbf{d}_2(a_2),\ldots,\mathbf{d}_n(a_n)). \tag{4.2.3}$$

Elimination of individual opportunities for gain,

$$p^0 \cdot [\mathfrak{v}_i(\mathbf{d}_i) - \mathfrak{v}_i] \leq 0, \; \forall \mathbf{d}_i, \forall i, \tag{4.2.4}$$

does not preclude the well-known possible existence of $\mathbf{d}$ with

$$p^0 \cdot [\mathfrak{v}_i(\mathbf{d}) - \mathfrak{v}_i] > 0, \forall i. \tag{4.2.5}$$

In $\mathscr{E}_{\text{ext}}$, externalities emerge as limitations on trading opportunities as defined by prices. The relevant indirect utlity function becomes

$$\overset{e}{\nu}{}_i^*(p; z_{-i}) = \inf_{z_i} \left\{ p \cdot z_i - \overset{e}{\nu}_i(z_i; z_{-i}) \right\}, \tag{4.2.6}$$

where $p \in \mathbb{R}^\ell$, as above, and the trades of others, $z_{-i} = \langle z_j \rangle_{j \neq i}$, are regarded as parameters of $\overset{e}{\nu}_i(\,\cdot\,; z_{-i})$. Externalities exists because markets/prices for z_{-i} are missing. They would be eliminated if markets were complete, i.e., if prices were of the same dimension as the relevant commodity space, $\mathbb{R}^{\ell\times n}$.

Definition 4: $(p^0,\langle z_i^0\rangle)$ is a price-taking equilibrium in $\mathscr{E}_{\text{ext}} = \langle \overset{e}{\nu}_i\rangle$ if

- $z_i^0 \in \partial \overset{e}{\nu}{}_i^*(p^0; z_{-i}^0), \quad \forall i.$
- $\sum_i z_i^0 = \mathbf{0}$

Like Nash (but not correlated) equilibrium in $\mathscr{G}$, the presence of externalities turns existence of price-taking equilibrium in $\mathscr{E}_{\text{ext}}$ into a fixed-point problem.

A play of a game is also 'public good' since, by construction, it is consumed by everyone. In Ostroy and Song [2009] an alternative conjugate duality resulting from the incentive constraints of correlated equilibrium characterizes departures from Lindahl-like efficiency pricing of public goods.

5 *Tâtonnement*

Under *tâtonnement* individuals respond to prices by registering their utility maximizing demands, after which the price of each commodity is adjusted in the obvious direction to reduce its excess demand. In *non*-quasilinear models of exchange, income effects of price changes imply that trade at disequilibrium prices would change what constitutes an equilibrium; hence, trade is not permitted until equilibrium prices are found. Nevertheless, even after precluding trade out of equilibrium, the presence of income effects in non-quasilinear models is known to create obstacles to convergence (Scarf [1960], Gale [1963]).

With quasilinearity, prices do not affect trading opportunities, Z_i; so there are no income effects associated with price changes. Allowing trade at disequilibrium prices would not change the definition of equilibrium, a property reminiscent of repeated play of a game. Similarities in convergence for economics and games, below, follow from similarities between the (p,z_i) duality for $\mathscr{E}$ and the (p,d_i) E-duality for $\mathscr{G}$.

5.1 *Tâtonnement* IN $\mathscr{E}$ WITH DIFFERENTIABILITY

With quasilinearity, concavity of utility functions guarantees that aggregate excess demands defined by

$$z(p) \in \partial V_{\mathscr{E}}^*(p)\big[=\sum_i \partial v_i^*(p)\big], \tag{5.1.1}$$

are the excess demands of an as-if single individual. Hence, the problem of finding equilibrium prices in $\mathscr{E}$ is the same as finding a minimum of the convex function $-V_{\mathscr{E}}^*$.

The traditional version of *tâtonnement* assumes each $\partial v_i^*(p)$ is a singleton, i.e., $z_i(p) = \nabla v_i^*(p)$, and therefore

$$z(p) = \sum_i z_i(p) = \nabla V_{\mathscr{E}}^*(p). \tag{5.1.2}$$

The *gradient algorithm* is

$$p^{t+1} = p^t - s[-z(p^t)]. \tag{5.1.3}$$

If $-V_{\mathscr{E}}^*(p^t) > \min_p -V_{\mathscr{E}}^*(p)$, the gradient of the convex differentiable function $-V_{\mathscr{E}}^*$ at p^t describes the direction of steepest descent, i.e.,

$$p^t \cdot [-z(p^t)] = p^t \cdot \nabla[-V_{\mathscr{E}}^*(p^t)] = \min_q \Big\{ \lim_{\lambda \searrow 0} \frac{-V_{\mathscr{E}}^*(p^t + \lambda q) - [-V_{\mathscr{E}}^*(p^t)]}{\lambda} : \|q\| = 1 \Big\}. \tag{5.1.4}$$

A well-known consequence of the gradient algorithm for a convex function is:

PROPOSITION 6: *(Tâtonnement* AS A GRADIENT ALGORITHM*) If $z(p^t) = \nabla V_{\mathscr{E}}^*(p^t)$ and s is sufficiently small, the gradient algorithm mimics the method of steepest descent for $-V^*$ and*

$$p^{t+1} = p^t - s[-z(p^t)] \Longrightarrow p^t \to p^0 \in P^0 \text{ and } z(p^t) \to \mathbf{0},$$

where $-V_{\mathscr{E}}^*(p^0) = \min_p -V_{\mathscr{E}}^*(p)$.

5.2 *Tâtonnement* IN $\mathscr{E}$ WITHOUT DIFFERENTIABILITY

The gradient algorithm implies $-V_{\mathscr{E}}^*(p^{t+1}) < -V_{\mathscr{E}}^*(p^t)$, i.e., it is always descending towards its goal. Without differentiability, this property need not hold. Nevertheless, convergence of prices can be achieved.

To accommodate non-differentiability, *tâtonnement* can be defined as a *subgradient algorithm*: starting with p^t at time t and $z(p^t) \in \partial V_{\mathscr{E}}^*(p^t)$, the auctioneer resets prices in the next period according to

$$p^{t+1} = p^t - s_t[-z(p^t)] = p^t + s_t z(p^t), \tag{5.2.1}$$

where $s_t > 0$, $t = 1,2,\ldots$. The value s_t is the step-size at t. Unlike the gradient algorithm where price changes are based only on excess demands, but are otherwise timeless, in the subgradient algorithm price adjustments to excess demands need to keep track of time. (The general condition is $\sum_t s_t = \infty$ and $\sum_t s_t^2 < \infty$.)

The following property of the subgradient algorithm established by Shor [1985, Theorem 2.2], as refined by Anstreicher and Wolsey [2009, Theorem 3], is:

PROPOSITION 7: *(Tâtonnement* AS A SUBGRADIENT ALGORITHM*)*
If $z(p^t) \in \partial V_{\mathscr{E}}(p^t)$ and $s_t = t^{-1}$, the subgradient algorithm for $\mathscr{E}$ yields:

$$p^{t+1} = p^t - s_t[-z(p^t)] \Longrightarrow p^t \to p^0 \in P^0.$$

Without differentiability of $V^*_{\mathscr{E}}$, convergence of prices does not suffice to imply convergence of quantities. (Recall the selection problem in Remark 3, above.) When $\partial V^*_{\mathscr{E}}(p)$ is not unique, the choice of an element in $\partial V^*_{\mathscr{E}}(p^t)$ determines $z(p^t)$. The selection may be such that there is $\alpha > 0$ and an infinite subsequence $\{t_k\}$ such that $\|z(p^{t_k}) - \mathbf{0}\| > \alpha$; i.e., while prices may be converging to their equilibrium values, quantities traded can be bounded away from being feasible. This is illustrated by the following.

EXAMPLE 2: (FAILURE OF CONVERGENCE OF EXCESS DEMANDS)
There is a single individual; hence the subscript i is omitted in the following. Let $Z = [-1,1]$ and $\nu(z) = z$, $z \in Z$. (Again, $\mathbf{0} = 0$.) It is readily established that

$$-\nu^*(p) = \max\{1 - p, p - 1\}.$$

Equilibrium price and quantity are $(p^0 = 1, z^0 = 0)$, while

$$\partial \nu^*(p) = \begin{cases} [-1,1] & \text{if } p = 1, \\ -1 & \text{if } p > 1, \\ 1 & \text{if } p < 1. \end{cases}$$

Failure of demand to be differentiable at $p = 1$ implies that $z(p)$ is discontinuous. When $p^1 = 1$, if the initial choice $z^1(p^1) \neq 0$, then $p^2 \neq 1$ and although $s_t \to 0$ implies $p^t \to 1$, $z^t(p^t)$ continually oscillates between −1 and 1. Similar conclusions would obtain if Z were the discrete set $\{-1,0,1\}$.

A refinement of the subgradient algorithm can be obtained by requiring differentiability only at equilibrium. Recalling the definition of differentiability of equilibrium for $\mathscr{E}$ as $\{\mathbf{0}\} = \partial V^*_{\mathscr{E}}(p^0)$, since the subgradient algorithm implies $p^t \to p^0$, and $z(p^t)$ has a convergent subsequence with limit $\overline{z}$, the well-known closedness of the subdifferential mapping implies $\overline{z} = \{\mathbf{0}\} = \partial V^*_{\mathscr{E}}(p^0)$. Therefore,

Proposition 8: *(* *Tâtonnement* with Differentiability at Equilibrium*)*
If $z(p^t) \in \partial V^*_{\mathscr{E}}(p^t)$, $p^t \to p^0$ and $\partial V^*_{\mathscr{E}}(p^0)$ is a singleton, then

$$p^{t+1} = p^t - t^{-1}[-z(p^t)] \Longrightarrow p^t \to p^0 \in P^0 \text{ and } z(p^t) \to \mathbf{0}.$$

The way to ensure differentiability at equilibrium is to assume it everywhere.

5.3 *Tâtonnement* in $\mathscr{E}$ with Indivisible Commodities

To eliminate reliance on differentiability, restrict feasible trades to be discrete, i.e.,

$$Z_i = \{z_i^1, z_i^2, \ldots, z_i^{K_i}\}.$$

Denote these restrictions on $\mathscr{E}$ as $\mathscr{E}_F$.

A possible consequence of discreteness (non-concavity) is that for some $z_i \in Z_i$ there may be no p for which it would be a utility maximizing choice, i.e.,

$$\{z_i\} \cap \partial v_i^*(p) = \emptyset, \forall p.$$

Discreteness will typically make the existence of price-taking equilibrium in $\mathscr{E}_F$ problematic. Nevertheless, there is always some choice consistent with price-taking maximization, i.e.,

$$Z_i \bigcap \partial v_i^*(p) \neq \emptyset, \forall p.$$

This suffices to define a subgradient algorithm.

The point of departure is that, like fictitious play, convergence can be defined by historical averages. Based on the history of past prices and corresponding utility maximizing choices, the historical average of excess demands at time t is

$$\overline{z}^t(p^t, p^{t-1}, \dots, p^1) := t^{-1}\Big(z(p^t) + z(p^{t-1}) + \cdots + z(p^1)\Big). \tag{5.3.1}$$

Although the historical average of excess demand is a function of current and past prices $(p^t, p^{t-1}, \dots, p^1)$, the previous period's historical average is a summary statistic of the past prices $(p^{t-1}, \dots, p^1)$. Therefore, the historical average of excess demand can be defined recursively by the following:

$$\overline{z}^t(p^t, \overline{z}^{t-1}) = t^{-1} z(p^t) + \big(1 - t^{-1}\big)\overline{z}^{t-1}. \tag{5.3.2}$$

The historical average of prices is

$$\overline{p}^t := t^{-1}\big(p^t + p^{t-1} + \cdots + p^1\big). \tag{5.3.3}$$

Modify the description of *tâtonnement* to be the *historical subgradient algorithm*,

$$p^{t+1} = \overline{p}^t + \overline{z}^t(p^t). \tag{5.3.4}$$

PROPOSITION 9: *(Tâtonnement* AS AN HISTORICAL SUBGRADIENT ALGORITHM)
In $\mathscr{E}_F$,

$$p^{t+1} = \overline{p}^t + \overline{z}^t(p^t) \Longrightarrow p^t \to p^0 \in P^0 \text{ and } \overline{z}^t(p^t) \to \mathbf{0}.$$

Proof. From Proposition 11, the subgradient algorithm

$$p^{t+1} = p^t + t^{-1} z(p^t) \Longrightarrow p^t \to p^0 \in P^0.$$

Restating the equality above for $t, t-1, \dots, 1$,

$$\begin{aligned} t\,p^{t+1} &= t\,p^t + z(p^t) \\ (t-1)p^t &= (t-1)p^{t-1} + z(p^{t-1}) \\ &\vdots \\ p^2 &= p^1 + z(p^1) \end{aligned}$$

Summing each side of these equalities,

$$t p^{t+1} + (t-1)p^t + \cdots + p^2 = t p^t + z(p^t) + (t-1)p^{t-1} + z(p^{t-1}) + \cdots + p^1 + z(p^1).$$

Cancelling yields

$$t p^{t+1} = p^t + p^{t-1} + \cdots + p^1 + z(p^t) + z(p^{t-1}) + \cdots z(p^1).$$

Therefore,

$$\begin{aligned} p^{t+1} &= t^{-1}\Big(\sum_{\tau=1}^{t} p^{\tau}\Big) + t^{-1}\Big(\sum_{\tau=1}^{t} z(p^{\tau})\Big) \\ &= \overline{p}^t + \overline{z}^t(p^t) \end{aligned}$$

Since $p^t \to p^0$, it follows that $\overline{p}^t \to p^0$; and therefore $\overline{z}^t(p^t) \to \mathbf{0}$. □

Proposition 9 says that without requiring concavity of utility functions, if the auctioneer keeps track of previous prices and excess demands and adjusts prices according to the historical subgradient algorithm, average prices and average excess demands converge to price-taking equilibrium.

5.3.1 Price-taking in $\mathscr{E}$ as a Linear Programming Problem

Each $z_i^k \in Z_i$ in $\mathscr{E}_F$ can be regarded as a unit level of 'activity' (i,k) having value $v_i(z_i^k)$. Achievement of the maximum gains in $\mathscr{E}_F$ can be formulated as the primal linear programming problem:

$$(LP_{\mathscr{E}_F}) \qquad \begin{aligned} &\max_{\langle \lambda_i^k \geq 0 \rangle} \sum_i \sum_k \lambda_i^k v_i(z_i^k) \\ &\text{subject to:} \\ &\sum_k \lambda_i^k = 1, \quad \forall i \\ &\sum_i \sum_k \lambda_i^k z_i^k = \mathbf{0}. \end{aligned}$$

The dual problem is:

$$(DLP_{\mathscr{E}_F}) \qquad \begin{aligned} &\min_{\langle\rho_i\rangle, p} \sum_i \rho_i \times 1 + p \cdot \mathbf{0} \\ &\text{subject to:} \\ &\rho_i + p \cdot z_i^k \geq v_i(z_i^k), \quad \forall k, \forall i. \end{aligned}$$

Since $\rho_i \geq v_i(z_i^k) - p \cdot z_i^k$ is to be minimized, it is readily established that the constraints of the dual can be incorporated into its objective function and the dual can be rewritten as

$$\min_p \sum_i \max_{z_i^k} \{v_i(z_i^k) - p \cdot z_i^k\} = \min_p - \sum_i v_i^*(p) = \min_p - V_{\mathscr{E}}^*(p). \tag{5.3.5}$$

In other words, a solution to (DLP$_{\mathscr{E}_F}$) is a p^0 such that $\mathbf{0} \in \partial V_{\mathscr{E}}^*(p^0)$.

The LP formulation can also be interpreted as concavification *over individuals* for a continuum version of $\mathscr{E}_F$ with finite types and unit mass of each type: prices p^0 and trades $\overline{z}_i^k$ represent existence of price-taking equilibrium, where $\overline{\lambda}_i^k$ is the fraction of type i trading $\overline{z}_i^k$.

The LP formulation can be extended from a finite number of activities $Z_i = \{z_i^1, \ldots, z_i^{K_i}\}$ for each i to *semi-infinite LP* where Z_i is compact convex subset, i.e., from $\mathscr{E}_F$ to $\mathscr{E}$. Other than having an infinite number of activities, the LP characterization of price-taking equilibrium also applies to $\mathscr{E}$.

REMARK 9: (CONVEXIFICATION AND REGRET) Anticipating the connection in Section 6, Hart and Mas-Collel's concept of regret ("if I knew then what I know now") applies to the historical subgradient algorithm for $\mathscr{E}_F$.

The average price in the historical subgradient algorithm for $\mathscr{E}_F$ converges to an optimal solution p^0 to the dual and the average excess demand converges to an optimal solution $\langle \overline{\lambda}_i^k \overline{z}_i^k \rangle$ to the primal where

$$\overline{z}(p^0) = \sum_i \sum_k \overline{\lambda}_i^k \overline{z}_i^k = \mathbf{0}. \tag{5.3.6}$$

The expression $\left(\sum_k \overline{\lambda}_i^k \overline{z}_i^k\right)$, where $\sum_k \overline{\lambda}_i^k = 1, \overline{\lambda}_i^k \geq 0$, calls attention to averaging as concavification of $\mathfrak{v}_i$ achieved *over time* in $\mathscr{E}_F$.

In the historical subgradient algorithm, the average of previous desired trades $\overline{z}^{t-1}$ are part of the price adjustment process. Therefore, utility maximizing choices at t can be compared to what has previously occurred. Rewrite

$$\overline{z}_i^{t-1} = t^{-1}\big(z_i(p^{t-1}) + z_i(p^{t-2}) + \cdots z_i(p^1) + z_i^0\big), \quad z_i^0 = \mathbf{0}, \tag{5.3.7}$$

as

$$\overline{z}_i^{t-1} = t^{-1}\sum_k \tau_i^k(\overline{z}_i^{t-1}) z_i^k, \tag{5.3.8}$$

where $\tau_i^k(\overline{z}_i^{t-1})$ is the number of times z_i^k is chosen in the sequence $\{z_i(p^\tau)\}_{\tau=0}^{t-1}$ defining $\overline{z}_i^{t-1}$. Hence, $t^{-1}\big[\sum_k \tau_i^k(\overline{z}_i^{t-1})\big] = 1$. In the following, dependence of τ_i^k on $\overline{z}_i^{t-1}$ is taken for granted.

Define regret for the historical subgradient algorithm in exchange as the difference between i's price-taking utility gain at t, given $\overline{p}^t$, if i could undo the past and the average utility of what i's previous trades would have yielded at $\overline{p}^t$.

$$r_i(\overline{p}^t, \overline{z}^{t-1}) := -\Big(v_i^*(\overline{p}^t) - t^{-1}\sum_k \tau_i^k\big[\overline{p}^t \cdot z_i^k - v_i(z_i^k)\big]\Big). \tag{5.3.9}$$

Since $v_i^*(p) = \inf_{z_i}\{p \cdot z_i - v_i(z_i)\}$, the minus sign makes regret non-negative. Note that

$$\sum_k \tau_i^k v_i(z_i^k) \le \widehat{v}_i\big(\sum_k \tau_i^k z_i^k\big). \tag{5.3.10}$$

The historical subgradient algorithm constructs the primal and dual solutions to the LP problem by continually reacting to regret by attempting to correct past 'mistakes;' i.e., convergence of regret to zero effectively constructs the relevant values of $\widehat{v}_i$. This can occur even when $z_i(p^t)$ does not converge, as in Example 2.

5.3.2 Price-taking in $\mathscr{G}$ as a Linear Programming Problem

Following the recipe for the LP formulation of $\mathscr{E}_F$, regard each $d_i^k \in D_i$ as a unit level of activity (i,k) having value $-\mathbf{v}_i(d_i^k)(=0)$. Achievement of equilibrium in $\mathscr{G}$ is described by the LP problem:

$$\text{(LP}_{\mathscr{G}}\text{)}\qquad \begin{aligned} &\max_{\langle\lambda_i^k\geq 0\rangle,\lambda_1\geq 0}\sum_i\sum_k\lambda_i^k[-\mathbf{v}_i(d_i^k)]+\lambda_1\\ &\text{subject to:}\\ &\qquad\sum_k\lambda_i^k=1,\forall i\\ &\qquad\sum_i\sum_k\lambda_i^k[-d_i^k]+\lambda_1\mathbf{1}=\mathbf{0}.\end{aligned}$$

The dual problem is:

$$\text{(DLP}_{\mathscr{G}}\text{)}\qquad \begin{aligned} &\min_{\langle\gamma_i\rangle,p}\sum_i\gamma_i\times 1+p\cdot\mathbf{0}\\ &\text{subject to:}\\ &\qquad\gamma_i-p\cdot d_i^k\geq-\mathbf{v}_i(d_i^k),\ \forall k,\forall i\\ &\qquad p\cdot\mathbf{1}\geq 1.\end{aligned}$$

Since $\gamma_i\geq p\cdot d_i^k-\mathbf{v}_i(d_i^k)$ is to be minimized, it is readily established that the constraints of the dual can be incorporated into its objective function and the dual can be rewritten as

$$\min_{\{p:p\cdot\mathbf{1}\geq 1\}}\sum_i\max_{d_i^k}\{p\cdot d_i^k-\mathbf{v}_i(d_i^k)\}=\min_{\{p:p\cdot\mathbf{1}\geq 1\}}\sum_i\mathbf{v}_i^*(p)=\min_{\{p:p\cdot\mathbf{1}\geq 1\}}\mathbf{V}_{\mathscr{G}}^*(p).$$

Analogously to (DLP$_{\mathscr{E}_F}$), a solution to (DLP$_{\mathscr{G}}$) is a p^0 such that $\mathbf{0}\in\partial\mathbf{V}_{\mathscr{G}}^*(p^0)$.

(DLP$_{\mathscr{G}}$) differs from (DLP$_{\mathscr{E}_F}$) due to the constraint on p that effectively stipulates prices must be non-zero. [In $\mathscr{E}$ (and $\mathscr{E}_F$), prices are $(p,1)\in\mathbb{R}^{\ell+1}$, where 1 is the price of the money commodity; hence prices are necessarily non-zero.] The Lagrangian variable associated with the constraint $p\cdot\mathbf{1}\geq 1$ is λ_1. Evidently, the objective function of DLP$_{\mathscr{G}}$ is minimized only when the price constraint is binding. Nevertheless, the value of the primal equals the value of the dual $(=0)$ only when $\lambda_1=0$. I.e., even though the price constraint is binding, a perturbation of the constraint to $p\cdot\mathbf{1}\geq\alpha$, $\alpha>0$ would not change the optimal value or the optimal solutions to (LP$_{\mathscr{G}}$).

5.4 *Tâtonnement* IN $\mathscr{G}$

The E-duality for $\mathscr{G}$ leads to the analog of *tâtonnement* with the qualification that, whereas prices $p \in \mathbb{R}^{\ell}$ are unrestricted for $\mathscr{E}$, in $\mathscr{G}$ $p \in \mathbb{R}^{A}$ is restricted to the probability mixtures P on A.

For $x \in \mathbb{R}^{A}$, let

$$Q(x \mid P) = \min \Big\{ p \in P : 2^{-1} \sum_{a} |x(a) - p(a)|^2 \Big\}; \tag{5.4.1}$$

and denote by $P_Q[x] \in P$ the argmin of $Q(x \mid P)$, i.e., the projection of x onto P as measured by the element of P that is the minimum quadratic distance from x. The mapping $P_Q : \mathbb{R}^{A} \to P$ is the continuously differentiable function,

$$P_Q(x) = x - \nabla_x Q(x \mid P). \tag{5.4.2}$$

At prices p^t, individuals report utility maximizing choices $d_i(p^t) \in \partial \mathbf{v}_i^*(p^t)$. Aggregating these choices as $d(p^t) = \sum_i d_i(p^t)$ to form $p^t - t^{-1} d(p^t)$, their sum does not typically belong to P. To convert this expression into next period's prices, the auctioneer revises p^t according to the *projected subgradient algorithm for* $\mathscr{G}$

$$p^{t+1} = P_Q\big[p^t - t^{-1} d(p^t)\big]. \tag{5.4.3}$$

Although price adjustments are based on aggregate excess demands, $d(p^t)$, they do not follow the simple rule that $p^{t+1}(a) - p^t(a)$ depends only on $d(p^t)(a)$. Under P_Q, adjustments to $p^t(a)$ typically depend on all excess demands $d(p^t)(b)$, $b \neq a$.

The projected subgradient algorithm leads to the same conclusions as the unconstrained version when the polyhedral convex function $\mathbf{V}^*_{\mathscr{G}}$ is constrained to lie in the simplex P.

PROPOSITION 10: *(Tâtonnement* AS A PROJECTED SUBGRADIENT ALGORITHM FOR $\mathscr{G}$*)*
If $d(p^t) \in \partial \mathbf{V}^*_{\mathscr{G}}(p^t)$*, the projected subgradient algorithm yields*

$$p^{t+1} = P_Q\big[p^t - t^{-1} d(p^t)\big] \Longrightarrow p^t \to p^0.$$

In parallel with Proposition 8 for $\mathscr{E}$, the projected subgradient algorithm can also achieve convergence of $d(p^t)$ if equilibria are differentiable.

PROPOSITION 11: *(Tâtonnement* WITH DIFFERENTIABILITY AT EQUILIBRIUM*)*
If $d(p^t) \in \partial \mathbf{V}^*_{\mathscr{G}}(p^t)$ *and* $\partial \mathbf{V}^*_{\mathscr{G}}(p^0)$ *is a singleton, then*

$$p^{t+1} = P_Q\big[p^t - t^{-1} d(p^t)\big] \Longrightarrow p^t \to p^0 \in P^0 \text{ and } d(p^t) \to \mathbf{0}.$$

Differentiability of equilibria is not a viable hypothesis for $\mathscr{G}$. However, the methods used to accommodate non-differentiability in $\mathscr{E}_F$ can be carried over to games.

The notation describing historical average prices $\overline{p}^t \in \mathbb{R}^\ell$ in $\mathscr{E}$ can also be used for $\overline{p}^t = t^{-1}\big(p^t + p^{t-1} + \cdots + p^1\big) \in \mathbb{R}^A$ in $\mathscr{G}$. The analog of historical average excess demands for $\mathscr{G}$ is

$$\overline{d}^t(p^t, p^{t-1}, \ldots, p^1) := t^{-1}\Big(d(p^t) + d(p^{t-1}) + \cdots + d(p^1)\Big). \tag{5.4.4}$$

Again, the historical deviation $\overline{d}^t$ can be defined recursively using the previous period's historical average as:

$$\overline{d}^t(p^t, \overline{d}^{t-1}) := t^{-1} d(p^t) + \big(1 - t^{-1}\big)\overline{d}^{t-1}. \tag{5.4.5}$$

The *projected historical subgradient algorithm for* $\mathscr{G}$ is,

$$p^{t+1} = P_Q\big[\overline{p}^t - \overline{d}^t(p^t)\big]. \tag{5.4.6}$$

PROPOSITION 12: *(Tâtonnement* AS A PROJECTED HISTORICAL SUBGRADIENT ALGORITHM FOR $\mathscr{G}$*)*
If $d(p^t) \in \partial \mathbf{V}^*_{\mathscr{G}}(p^t)$,

$$p^{t+1} = P_Q\big[\overline{p}^t - \overline{d}^t(p^t)\big] \Longrightarrow p^t \to p^0 \text{ and } \overline{d}^t(p^t) \to \mathbf{0}.$$

Proposition 12 follows from 10 for $\mathscr{G}$ as Proposition 9 follows from 7 for $\mathscr{E}$.

6 AUTONOMOUS PRICE ADJUSTMENT VIA FICTITIOUS PLAY

The previous section demonstrated convergence in games from the perspective of price adjustment in economics. Positive conclusions were obtained, but they required giving up the following appealing properties of the standard model of *tâtonnement* with differentiability.

(1) Substituting $d(p) = \sum_i d_i(p)$ for $z(p) = \sum_i z_i(p)$ in the definition of aggregate excess demands, the utility consequences of trades/deviations were substituted for the commodity descriptions of trades; hence, price changes depend explicitly on utilities.

(2) Substituting the projection mapping $p^{t+1} = P_Q\big[p^t - s_t[d^t]\big]$ for $p^{t+1} = p^t - s_t z^t$, adjustments to $p^t(a)$ depend on all the elements of $d \in \mathbb{R}^A$ rather than being separately implementable, commodity by commodity, as they are in *tâtonnement*.

(3) The subgradient algorithm required keeping track of time; and the historical subgradient algorithm required keeping track of historical averages.

(3) is attributable to the absence of differentiability, either in $\mathscr{E}$ or $\mathscr{G}$. Both of these temporal qualifications are also present in fictitious play. Nevertheless, fictitious play obviates the need for (1) and (2), thereby avoiding the auctioneer-directed procedure in the standard model of *tâtonnement*. The following reformulates the E-duality to be amenable to fictitious play.

6.1 From the E- to the F-Duality

From $\mathbf{d}_i : A_i \to A_i$, let $\mathbf{d}_i(a_i) = b_i$ and define

$$\mathfrak{p}_i(p)(b_i \mid a_i) := \sum_{a_{-i}} p(a_i,a_{-i})[\mathfrak{v}_i(b_i,a_{-i}) - \mathfrak{v}_i(a_i,a_{-i})]. \tag{6.1.1}$$

Interpreting $[\mathfrak{v}_i(b_i,a_{-i}) - \mathfrak{v}_i(a_i,a_{-i})]$ as the marginal utility at a_i of switching to b_i, $\mathfrak{p}_i(p)(b_i \mid a_i)$ is p-weighted marginal utility of such a change. Therefore, $\mathfrak{p}_i(p)(a_i \mid a_i) = 0$. Only those a_i in the support of p can have non-zero p-weighted marginal utilities, i.e.,

$$p(a_i) = \sum_{a_{-i}} p(a_i,a_{-i}) = 0 \Longrightarrow \mathfrak{p}_i(p)(b_i \mid a_i) = 0, \forall b_i. \tag{6.1.2}$$

From a_i, the vector

$$\mathfrak{p}_i(p)[a_i] = \langle \mathfrak{p}_i(p)(b_i \mid a_i) \rangle_{b_i \in A_i} \in \mathbb{R}^{A_i}, \tag{6.1.3}$$

are the possible p-weighted marginal utilities, one for each $b_i \in A_i$. For all $a_i \in A_i$, the p-weighted marginal utilities are denoted by $\mathfrak{p}_i(p) = \langle \mathfrak{p}_i(p)[a_i] \rangle_{a_i \in A_i} \in \mathbb{R}^{A_i} \times \mathbb{R}^{A_i}$.

Let $\mathfrak{e}_i : A_i \to \mathbb{R}^{A_i}$ where for each a_i, $\mathfrak{e}_i(a_i) = \mathbf{1}_{b_i}$ for some $b_i \in A_i$. The deviation $b_i = \mathbf{d}_i(a_i)$ is now represented as an element in $\mathbb{R}^{A_i}$. Evidently, each $\mathfrak{e}_i$ corresponds to exactly one $\mathbf{d}_i : A_i \to A_i$. Analogously, $\mathbf{d}_i^{\text{ID}}$ becomes $\mathfrak{e}_i^{\text{ID}}$, where $\mathfrak{e}_i^{\text{ID}}(a_i) = \mathbf{1}_{a_i}$ for all a_i. Extending $\mathfrak{e}_i$, let $\mathfrak{z}_i : A_i \to \mathbb{R}^{A_i}$, where

$$\mathfrak{z}_i[a_i] = \langle \mathfrak{z}_i(b_i \mid a_i) \rangle_{b_i \in A_i} \in \mathbb{R}^{A_i}, \tag{6.1.4}$$

for all a_i.

The F-duality for $\mathcal{G}$ is

$$(\mathfrak{p}_i[a_i], \mathfrak{z}_i[a_i]) \in \mathbb{R}^{A_i} \times \mathbb{R}^{A_i}$$

for each a_i, or $(\mathfrak{p}_i, \mathfrak{z}_i) \in \left(\mathbb{R}^{A_i} \times \mathbb{R}^{A_i}\right) \times \left(\mathbb{R}^{A_i} \times \mathbb{R}^{A_i}\right)$ for all a_i. Formally, prices in the F-duality are $\mathfrak{p}_i[a_i] \in \mathbb{R}^{A_i}$, whether or not they are derived from p.

6.2 Utility Maximization

Feasible trades from a_i are circumscribed by

$$\mathbf{u}_i(\mathfrak{z}_i[a_i]) = \begin{cases} 0 & \text{if } \mathfrak{z}_i[a_i] \in \Delta[A_i], \\ \infty & \text{otherwise;} \end{cases} \tag{6.2.1}$$

with the resulting conjugate function

$$\mathbf{u}_i^*(\mathfrak{p}_i)[a_i] = \sup_{\mathfrak{z}_i[a_i]} \left\{ \mathfrak{p}_i[a_i] \cdot \mathfrak{z}_i[a_i] - \mathbf{u}_i(\mathfrak{z}_i[a_i]) \right\} \tag{6.2.2}$$

To establish equivalence with the E-duality, its suffices to observe that when $\mathfrak{e}_i(a_i) = 1_{b_i}$, i.e., $\mathbf{d}_i(a_i) = b_i$,

$$\mathfrak{p}_i(p)(b_i \mid a_i) = \mathfrak{p}_i(p)[a_i] \cdot \mathfrak{e}_i[a_i] = p \cdot d_i(\mathbf{d}_i(a_i)). \tag{6.2.3}$$

Hence, when $\mathfrak{e}_i \sim \mathbf{d}_i$,

$$\mathfrak{p}_i(p) \cdot \mathfrak{e}_i = \sum_{a_i} \mathfrak{p}_i(p)[a_i] \cdot \mathfrak{e}_i[a_i] = \sum_{a_i} \sum_{a_{-i}} p(a_i, a_{-i}) \cdot d_i(\mathbf{d}_i(a_i), a_{-i})] = p \cdot d_i(\mathbf{d}_i) \tag{6.2.4}$$

When $\mathfrak{p}_i[a_i]$ is derived from p as $\mathfrak{p}_i(p)[a_i]$, overall utility maximization at $\mathfrak{p}_i = \mathfrak{p}_i(p)$ is

$$\begin{aligned}\mathbf{u}_i^*(\mathfrak{p}_i(p)) &:= \sum_{a_i} p(a_i)\mathbf{u}_i^*(\mathfrak{p}_i(p))[a_i] \\ &= \sum_{a_i} \sup_{\mathfrak{e}_i[a_i]} p(a_i)\{\mathfrak{p}_i(p)(a_i)\cdot \mathfrak{e}_i[a_i] - \mathbf{u}_i(\mathfrak{e}_i[a_i])\} \\ &= \sup_{d_i}\{p\cdot d_i - \mathbf{v}_i(d_i)\} \\ &= \mathbf{v}_i^*(p)\end{aligned} \tag{6.2.5}$$

The first equality highlights the fact that overall maximization can be separated into maximization with respect to each a_i (in the support of p). This property is also a feature of the E-duality duality, but it was not exploited because the emphasis was on establishing similarities with $\mathscr{E}$ where separability is not typical. The second equality — that each $\mathbf{u}_i^*(\mathfrak{p}_i(p))[a_i]$ can be achieved without randomization — is mirrored in the third and fourth equalities previously employed for the E-duality.

The essential difference between the two dualities is that instead of the utility consequences of deviations, $d_i(\mathbf{d}_i)$, playing the role of quantities, quantities are represented by $\mathfrak{e}_i$ (or its randomized counterpart $\mathfrak{z}_i$) which, like z_i in $\mathscr{E}$, are not defined in units of utility. And, instead of probabilities directly playing the role of prices, the (vector of) marginal-utility consequences of those probabilities, $\mathfrak{p}_i(p)[a_i]$, represent prices to i. Information from $\mathfrak{v}_i$ defining $d_i(a_i) = \mathfrak{v}_i(\mathbf{d}_i(a_i), a_{-i}) - \mathfrak{v}_i(a_i, a_{-i})$ in the E-duality is used in F-duality to define $\mathfrak{p}_i(p)(\mathbf{d}_i(a_i)\,|\,a_i)$; but *the latter transformation is achieved in a decentralized way* since the conversion of p into $\mathfrak{p}_i(p)(\mathbf{d}_i(a_i)\,|\,a_i)$ is made only by i.

Therefore, by a slight abuse of notation, $\mathbf{u}_i^*(\mathfrak{p}_i(p))$ can be written in more abbreviated form as $\mathbf{u}_i^*(p)$, where the subscript i incorporates information about $\mathfrak{v}_i$ that translates p into $\mathfrak{p}_i(p)$. Note, however, that for the definition of $\partial\mathbf{u}_i^*(\cdot)[a_i]$ only the vector $\mathfrak{p}_i[a_i]$, ignoring its possible origins in p, is relevant. Thus, when $\mathfrak{p}_i[a_i] = \mathfrak{p}_i(p)[a_i]$, $\partial\mathbf{u}_i^*(\mathfrak{p}_i(p))[a_i]$ is written as

$$\partial_{\mathfrak{p}_i}\mathbf{u}_i^*(p)[a_i] := \Big\{\mathfrak{z}_i[a_i] : \big(\mathfrak{p}_i'[a_i] - \mathfrak{p}_i(p)[a_i]\big)\cdot \mathfrak{z}_i[a_i] \le \mathbf{u}_i^*(\mathfrak{p}_i')[a_i] - \mathbf{u}_i^*(\mathfrak{p}_i(p))[a_i], \forall \mathfrak{p}_i'[a_i] \in \mathbb{R}^{A_i}\Big\}. \tag{6.2.6}$$

It is readily confirmed that (individually perceived) expected profits are non-negative, i.e.,

$$p(a_i)\mathbf{u}_i^*(p)[a_i] \ge 0, \quad \forall p \in P, \forall a_i, \forall i. \tag{6.2.7}$$

Correlated equilibrium is characterized by the complementary slackness condition:

$$p^0 \in P^0 \Longleftrightarrow p^0(a_i)\mathbf{u}_i^*(p^0)[a_i] = 0, \quad \forall a_i, \forall i. \tag{6.2.8}$$

In economic terminology, equilibrium prices eliminate profit opportunities.

REMARK 10: (UTILITY MAXIMIZATION WITH AND WITHOUT INDEPENDENT PROBABILITIES) Let $q = \langle q_i \rangle \in \times_i \Delta[A_i] = Q$ represent independent probabilities, a non-convex subset of P, and $p \in P \backslash Q$. A well-known property (associated with Nash equilibrium) is: i would be indifferent between $a_i \neq a_i'$ in the support of q, i.e., $\mathbf{u}_i^*(q)[a_i] = \mathbf{u}_i^*(q)[a_i']$. However, if a_i and a_i' are in the support of p, then $\mathbf{u}_i^*(p)[a_i]$ need not equal $\mathbf{u}_i^*(p)[a_i']$ and utility maximizing choices can differ, i.e., $\partial_{\mathfrak{p}_i}\mathbf{u}_i^*(p)[a_i] \cap \partial_{\mathfrak{p}_i}\mathbf{u}_i^*(p)[a_i'] = \emptyset$. This distinction is relevant for fictitious play.

6.3 UTILITY MAXIMIZATION WITH QUADRATIC COSTS

The polyhedral property of $\mathbf{u}_i^*$ implies that $\partial_{\mathfrak{p}_i}\mathbf{u}_i^*$ is not singleton-valued, i.e., not differentiable. A standard method of converting a non-differentiable into a differentiable optimization problem is to introduce a quadratic penalty/cost. Add

$$Q_i(\mathfrak{z}_i[a_i]) := 2^{-1} \sum_{b_i \neq a_i} |\mathfrak{z}_i(b_i \mid a_i)|^2 \tag{6.3.1}$$

to define the modification of $\mathbf{u}_i$ as

$$\mathbf{U}_i(\mathfrak{z}_i[a_i]) := \mathbf{u}_i(\mathfrak{z}_i[a_i]) + Q_i(\mathfrak{z}_i[a_i]) \tag{6.3.2}$$

Thus, $\mathbf{U}_i(\mathfrak{z}_i[a_i]) = 0$ if and only if $\mathfrak{z}_i[a_i] = \mathfrak{e}_i^{\mathrm{ID}}[a_i]$. The conjugate of $\mathbf{U}_i$ is

$$\mathbf{U}_i^*(\mathfrak{p}_i)[a_i] = \sup_{\mathfrak{z}_i[a_i]} \left\{\mathfrak{p}_i[a_i] \cdot \mathfrak{z}_i[a_i] - \mathbf{U}_i(\mathfrak{z}_i[a_i])\right\} \tag{6.3.3}$$

Without loss of generality, $\mathfrak{v}_i$ can be scaled so that

$$\max_{a,b \in A} \{\mathfrak{v}_i(b) - \mathfrak{v}_i(a), 0\} < 1, \quad \forall i. \tag{6.3.4}$$

A consequence of the rescaling is that when $p \in P$ and $\mathfrak{p}_i[a_i] = \mathfrak{p}_i(p)[a_i]$,

$$\mathfrak{p}_i(p)^+(b_i \mid a_i) = \begin{cases} \max\{\mathfrak{p}_i(p)(b_i \mid a_i), 0\} < p(a_i) & \text{if } p(a_i) > 0, \\ 0 & \text{if } p(a_i) = 0. \end{cases} \tag{6.3.5}$$

Write $\mathfrak{z}_i(p)[a_i] = \nabla_{\mathfrak{p}_i} \mathbf{U}_i^*(p)[a_i]$. Therefore,

$$\mathbf{U}_i^*(p)[a_i] = \mathfrak{p}_i(p)[a_i] \cdot \mathfrak{z}_i(p)[a_i] - \mathbf{U}_i(\mathfrak{z}_i(p)[a_i]) \tag{6.3.6}$$

From $\nabla_{\mathfrak{p}_i} \mathbf{U}_i^*(p)[a_i]$, the first-order necessary and sufficient conditions for utility maximization with respect to each possible deviation from a_i is:

$$\underbrace{\mathfrak{p}_i^+(p)(b_i \mid a_i)}_{\text{p-weighted marginal utility}} = \underbrace{\mathfrak{z}_i(p)(b_i \mid a_i)}_{\text{marginal cost}}, \quad b_i \neq a_i. \tag{6.3.7}$$

Ignoring costs, expected utility gains at p for i at a_i is $|\mathfrak{p}_i^+(p)[a_i]| := \sum_{b_i} \mathfrak{p}_i^+(p)(b_i \mid a_i)$. Scaling of utilities and quadratic costs implies

$$1 - |\mathfrak{p}_i^+(p)[a_i]| = \mathfrak{z}_i(p)(a_i \mid a_i) > 0, \tag{6.3.8}$$

guaranteeing that $\mathfrak{z}_i(a_i) \in \Delta[A_i]$. Hence, the constraint $\sum_{b_i \in A_i} \mathfrak{z}_i(b_i \mid a_i) = 1$ is never binding. From (6.3.7) and (6.3.8),

$$\mathfrak{z}_i(p)[a_i] = \mathfrak{p}_i^+(p)[a_i] + \big(1 - |\mathfrak{p}_i^+(p)[a_i]|\big)\mathfrak{e}_i^{\text{ID}}[a_i]. \tag{6.3.9}$$

Hence,

$$\mathbf{U}_i^*(p)[a_i] = 2^{-1} \nabla_{\mathfrak{p}_i} \mathbf{U}_i^*(p)[a_i] \cdot \mathfrak{p}_i^+(p)[a_i] = 2^{-1} |\mathfrak{p}_i^+(p)[a_i]|^2. \tag{6.3.10}$$

By construction, $\mathbf{U}_i^*(p)[a_i] \leq \mathbf{u}_i(p)[a_i]$; however, adding quadratic costs does not change the set of equilibrium prices: p^0 is a correlated equilibrium if and only if $\mathfrak{e}_i^{\text{ID}}[a_i] = \nabla_{\mathfrak{p}_i} \mathbf{U}_i^*(p^0)[a_i]$ for all a_i in the support of p^0; or

$$p^0(a_i) \mathbf{U}_i^*(p^0)[a_i] = 0 \Longleftrightarrow p^0(a_i) \mathbf{u}_i^*(p^0)[a_i] = 0, \quad \forall a_i, \forall i. \tag{6.3.11}$$

REMARK 11: (REGRET) When p represents the frequencies of past choices, Hart and Mas-Colell [2000] describe $\mathfrak{p}_i^+(p)(b_i \mid a_i)$ as i's *regret* in previously having chosen a_i when i could have chosen b_i; and $\mathfrak{z}_i(p)[a_i] = \nabla_{\mathfrak{p}_i}\mathbf{U}_i^*(p)[a_i]$ are choices exhibiting *regret-matching*. (See, below.) Regret-matching is an adaptive procedure for dealing with a changing environment in which individuals can be regarded as learning by correcting past 'mistakes.' Consistent with the goal of calling attention to similarities between game theory and general equilibrium, the formulation, above, chararacterizes regret-matching as a smoothing, via trading costs, of price-taking utility maximization.

REMARK 12: (SMOOTH FICTITIOUS PLAY) Fudenberg and Levine [1998] adopt a related utility maximizing approach in which choices are smoothed by adding costs. Their cost functions are such that individuals always choose to be in the interior of $\Delta[A_i]$. To illustrate using their specific example, rewritten to conform to current conventions, the cost of $\mathfrak{z}_i[a_i]$ is

$$C_i(\mathfrak{z}_i[a_i];\lambda) = \begin{cases} \lambda \sum_{b_i \in A_i} \mathfrak{z}_i(b_i \,|\, a_i) \log \mathfrak{z}_i(b_i \,|\, a_i) & \text{if } \mathfrak{z}_i[a_i] \in \Delta[A_i], \\ \infty & \text{otherwise} \end{cases} \tag{6.3.12}$$

where $\lambda > 0$. The resulting conjugate is

$$C_i^*(p;\lambda)[a_i] = \sup_{\mathfrak{z}_i[a_i]} \left\{ \mathfrak{p}_i(p) \cdot \mathfrak{z}_i[a_i] - C_i(\mathfrak{z}_i[a_i];\lambda) \right\}. \tag{6.3.13}$$

The (smooth) best-response function is

$$\nabla_{\mathfrak{p}_i} C_i^*(p;\lambda)[a_i] = \left\langle \mathfrak{z}_i(p;\lambda)(b_i \,|\, a_i) \right\rangle = \left\langle \frac{\exp\left(\lambda^{-1}\mathfrak{p}_i(p)(b_i \,|\, a_i)\right)}{\sum_{b_i' \in A_i} \exp\left(\lambda^{-1}\mathfrak{p}_i(p)(b_i' \,|\, a_i)\right)} \right\rangle \gg \mathbf{0}. \tag{6.3.14}$$

In overall agreement with $\mathfrak{z}_i(p)[a_i] = \nabla_{\mathfrak{p}_i}\mathbf{U}_i^*(p)[a_i]$, the larger the value of $\mathfrak{p}_i(p)(b_i \,|\, a_i)$, the larger is $\mathfrak{z}_i(p;\lambda)(b_i \,|\, a_i)$; and the gap between $\nabla_{\mathfrak{p}_i}\mathbf{U}_i^*(p)[a_i]$ and $\nabla_{\mathfrak{p}_i} C_i^*(p;\lambda)[a_i]$ narrows as λ decreases.

6.4 CONVERGENCE

6.4.1 Randomized Fictitious Play

The choices $a = \langle a_i \rangle \in A$ can be represented as

$$\mathbf{1}_a = \times_i \mathbf{1}_{a_i} \in \mathbb{R}^A. \tag{6.4.1}$$

The average of the sequence $\{\mathbf{1}_{a^\tau} : \tau = 1, \ldots, t\}$ is

$$\overline{p}^t = t^{-1}\Big(\sum_{\tau=1}^{t} \mathbf{1}_{a^\tau}\Big). \tag{6.4.2}$$

The average is updated as a convex combination of $\overline{p}^t$ and $\mathbf{1}_{a^{t+1}}$, i.e.,

$$\overline{p}^{t+1} = \tfrac{t}{t+1}\overline{p}^t + \tfrac{1}{t+1}\mathbf{1}_{a^{t+1}}. \tag{6.4.3}$$

Fictitious play adopts the price-taking hypothesis that the current average, $\overline{p}^t$, will continue. Without costs, i's best response at a_i^t and $\overline{p}^t$ is an element $\mathbf{1}_{a_i^{t+1}} = \mathfrak{e}_i[a_i^t](\overline{p}^t) \in \partial_{\mathfrak{p}_i}\mathbf{u}_i^*(\overline{p}^t)[a_i^t]$. Therefore,

$$\mathbf{1}_{a^{t+1}} = \times_i \mathbf{1}_{a_i^{t+1}} \in \times_i \mathfrak{e}_i(\overline{p}^t)[a_i^t],$$

where $a^{t+1} = \langle a_i^{t+1}\rangle$. The outcome is well-defined whenever $\partial_{\mathfrak{p}_i}\mathbf{u}_i^*(\overline{p}^t)[a_i^t]$ is a singleton for all i; otherwise, there is a selection problem.

With costs, maximization leads to (uniquely defined) randomized fictitious play, where

$$\mathfrak{z}_i(\overline{p}^t)[a_i^t] = \langle \mathfrak{z}_i(\overline{p}^t)(a_i^{t+1} \,|\, a_i^t)\rangle = \nabla_{\mathfrak{p}_i}\mathbf{U}_i^*(\overline{p}^t)[a_i^t], \tag{6.4.4}$$

is the vector of (cost-adjusted) utility maximizing choices of i at $a_i = a_i^t$ when facing prices $p = \overline{p}^t$.

The state at t is defined by the most recent realization, $\mathbf{1}_{a^t} = \times_i \mathbf{1}_{a_i^t}$, and the history of those realizations, $\overline{p}^t$. The probability of $\mathbf{1}_{a^{t+1}} = \times_i \mathbf{1}_{a_i^{t+1}}$ depends on the state at t via the product of the probabilities $\times_i \mathfrak{z}_i(\overline{p}^t)[a_i^t] \in \times_i \Delta[A_i]$ as

$$\text{Prob}\big\{\mathbf{1}_{a^{t+1}} \,|\, \mathbf{1}_{a^t}; \overline{p}^t\big\} = \times_i \mathfrak{z}_i(\overline{p}^t)(a_i^{t+1} \,|\, a_i^t). \tag{6.4.5}$$

This formulation of randomized fictitious play has been shown to imply the following probabilisitic conclusions on convergence to correlated equilibrium.

PROPOSITION 13: *(Hart and Mas-Colell [2000])*
The model of randomized fictitious play defined by (6.4.1–5) implies

$$\forall \epsilon > 0, \quad \lim_{t\to\infty} Prob\left\{\mathbf{U}_i^*(\overline{p}^t)[a_i^t] > \epsilon\right\} = 0, \forall i.$$

6.4.2 Randomized Fictitious Play in Populations

Given $\overline{p}^t$ and $\mathbf{1}_{a^t}$, $\overline{p}^{t+1}$ is a random variable defined by (6.4.5). The expectation of $\overline{p}^{t+1}$,

$$\mathbf{E}(\overline{p}^{t+1}\,|\,\overline{p}^t;\mathbf{1}_{a^t}) = \tfrac{t}{t+1}\overline{p}^t + \tfrac{1}{t+1}\sum_{a^{t+1}=\langle a_i^{t+1}\rangle}\mathbf{1}_{a^{t+1}} \times \text{Prob}\big\{\mathbf{1}_{a^{t+1}}\,|\,\mathbf{1}_{a^t};\overline{p}^t\big\}, \tag{6.4.6}$$

is also random variable depending on the realization $\mathbf{1}_{a^t}$.

When a finite number of types of individuals make probabilistic choices, a continuum interpretation, called the *population model of randomized fictitious play*, can be used to conclude that each type's randomized choice is achieved with certainty in the population as a whole. In the population model, the evolution of prices is deterministic.

Assuming mass 1 of each i, $p \in P$ describes an existing distribution of the population, where

$$\Big\langle p(a_i) = \sum_{a_{-i}} p(a_i, a_{-i})\Big\rangle \in \Delta[A_i] \tag{6.4.7}$$

is the distribution of individuals of type i. At p, if each a_i in the support of p chooses to make the changes $\mathfrak{z}_i[a_i] \in \Delta[A_i]$, the resulting distribution of the population is

$$p[\times_i \mathfrak{z}_i] := \sum_{a=\langle a_i\rangle} p(a)[\times_i \mathfrak{z}_i][a_i] \in P, \tag{6.4.8}$$

the population weighted average of those changes. While each $[\times_i \mathfrak{z}_i][a_i] \in \times_i \Delta[A_i]$ represents independently chosen randomizations, their convex combination (6.4.8) may not belong to $\times_i \Delta[A_i]$, i.e., may exhibit correlation. The population distribution does not change when $p = p[\times_i \mathfrak{e}_i^{\text{ID}}]$.

The average in the population version of the model is denoted by $\overline{\mathbf{p}}^t$ and is based on the sequence of *realized populations*, i.e.,

$$\overline{\mathbf{p}}^t = t^{-1}\Big(\sum_{\tau=1}^{t} p^t\Big), \tag{6.4.9}$$

where p^{t+1} is determined by $\overline{\mathbf{p}}^t$ and p^t as

$$p^{t+1} = p^t[\times_i \mathfrak{z}_i(\overline{\mathbf{p}}^t)]. \tag{6.4.10}$$

Note that the description of utility maximizing behavior $\times_i \mathfrak{z}_i(\cdot)(b_i\,|\,a_i)$, where $\langle \mathfrak{z}_i(\cdot)(b_i\,|\,a_i)\rangle =$

$\nabla_{\mathfrak{p}_i}\mathbf{U}_i^*(\cdot)[a_i]$, from randomized fictitious play is used in the population version.

The state of the population model at t is $(\overline{\mathbf{p}}^t, p^t)$. Hence, for those a_i such that $p^t(a_i) > 0$, their choices are $\mathfrak{z}_i(\overline{\mathbf{p}}^t)[a_i] = \langle \mathfrak{z}_i(\overline{\mathbf{p}}^t)(b_i \,|\, a_i)\rangle$. Updating of the average is

$$\begin{aligned}\overline{\mathbf{p}}^{t+1} &= \tfrac{t}{t+1}\overline{\mathbf{p}}^t + \tfrac{1}{t+1}p^{t+1}\\ &= \tfrac{t}{t+1}\overline{\mathbf{p}}^t + \tfrac{1}{t+1}p^t[\times_i \mathfrak{z}_i(\overline{\mathbf{p}}^t)].\end{aligned} \tag{6.4.11}$$

In contrast to (6.4.6) for randomized fictitious play, in the population model the realized and expected values of $\overline{\mathbf{p}}^{t+1}$ coincide, i.e.,

$$\begin{aligned}\mathbf{E}(\overline{\mathbf{p}}^{t+1} \,|\, \overline{\mathbf{p}}^t; p^t) &= \tfrac{t}{t+1}\overline{\mathbf{p}}^t + \tfrac{1}{t+1}\mathbf{E}(p^{t+1} \,|\, \overline{\mathbf{p}}^t; \mathbf{1}_{a^t})\\ &= \tfrac{t}{t+1}\overline{\mathbf{p}}^t + \tfrac{1}{t+1}\sum_a p^t(a^t) \times_i \mathfrak{z}_i(\overline{\mathbf{p}}^t)[a^t]\\ &= \overline{\mathbf{p}}^{t+1}\end{aligned} \tag{6.4.12}$$

Adapting arguments from the demonstration of convergence in the previous conclusion:

PROPOSITION 14: *(*POPULATION CONVERGENCE*)*
The model of randomized fictitious play in populations defined by (6.4.7–11) implies

$$\lim_{t\to\infty}\sum_{a_i} p^t(a_i)\mathbf{U}_i^*(\overline{\mathbf{p}}^t)[a_i] = 0, \quad \forall i.$$

A detailed proof is available as an online appendix.[1]

REMARK 13: (METRIC AND PSEUDO-METRIC CONVERGENCE) *Tâtonnement* results for $\mathscr{E}$ and $\mathscr{G}$ in Section 5 follow the standard definition of convergence of prices: the sequence $\{p^t\}$ converges if there is a p such that the norm, hence metric, distance $\|p^t - p\| \to 0$. A *pseudo-metric* satisfies the conditions of a metric except the distance between distinct points can be zero.

The F-duality calls attention to the minimizing condition for equilibrium prices as the elimination of profit opportunities $p^0(a_i)\mathbf{U}_i^*(p^0)[a_i] = 0, \forall a_i, \forall i$. A relevant measure of disequilibrium is the expected utility gains associated with prices in $\mathscr{G}$,

$$\mathbf{U}^*(p) = \sum_i \sum_{a_i} p(a_i)\mathbf{U}_i^*(p)[a_i] \geq 0. \tag{6.4.13}$$

[1] `https://goo.gl/rzSv9G`.

Expected utility gains can be used to define

$$d_{\mathbf{U}^*}(p_1,p_2) = |\mathbf{U}^*(p_1) - \mathbf{U}^*(p_2)|, \tag{6.4.14}$$

as a pseudo-metric for P in which distance between p_1 and p_2 is measured by the difference in equivalence classes

$$P^{\alpha} = \{p : \mathbf{U}^*(p) = \alpha\}$$

to which they belong. Convergence of p^t to p in the pseudo-metric requires only that $d_{\mathbf{U}^*}(p^t,p) \to 0$.

Population convergence in Proposition 14 is with respect to the pseudo-metric, i.e., $d_{\mathbf{U}^*}(\overline{\mathbf{p}}^t, P^0) \to 0$. If $P^0 = \{p^0\}$ metric and pseudo-metric convergence coincide; but for $\mathscr{G}$, P^0 is a convex set that is not typically a singleton. Although pseudo-metric convergence stipulates that expected gains are converging to zero for all members of the population, the larger is P^0 the less that can be said about how this translates to which elements of P^0 are being approached and, hence, the resulting distribution of utility gains. The conclusion in Proposition 13 for randomized fictitious play is a probabilistic version of convergence in the pseudo-metric.

REMARK 14: (POPULATION CONVERGENCE) The population setting is especially suited to a price-taking interpretation because a single individual's response does not change the distribution. With a finite number of individuals, if any one were to change their behavior, e.g., by misrepresenting their preferences and therefore their responses, that individual might change the equilibrium in their favor. Such favorable misrepresentation is well-known when $\mathscr{E}$ describes a model with a finite number of individuals (rather than a finite number of types).

Population models are used in evolutionary games, e.g., Friedman [1991], Samuelson [1997], where individual behavior may be fixed and transition probabilities are based on measures of relative fitness. In the population model here, 'fixed' behavior is individuals' reactions to currently perceived gains as determined by $\nabla_{\mathfrak{p}_i} \mathbf{U}_i^*(\overline{\mathbf{p}}^t)$ that determines transition probabilities $p^{t+1} = p^t[\times_i \mathfrak{z}_i(\overline{\mathbf{p}}^t)]$.

7 Concluding Remark

Connections between game theory and general equilibrium have employed cooperative or non-cooperative solution concepts from games to rationalize Walrasian equilibrium. The direction emphasized here is from economics (Walrasian equilibrium) to games (correlated equilibrium). But the intent is not to rationalize the latter by the former. Rather it is to view correlated equilibrium as a modification of the Walrasian approach to general equilibrium that retains much of its formal and informal structure.

The informational setting underlying price-taking in economics is: individuals make decisions knowing which commodities can be traded, their prices, and how their choices affect their utilities, without knowing the utilities of others. A similar informational setting consistent with (convergence to) correlated equilibrium is: individual i knows the choices A_i, the choices available to others, A_{-i}, prices (in the F-duality, frequencies of previous choices), and how those choices affect i's utility, $\mathfrak{v}_i$, without requiring knowledge of $\mathfrak{v}_j$, $j \neq i$.

A formal property of the modification from $\mathscr{E}$ to $\mathscr{G}$ is:

> Instead of the default description of economic interactions defined by trade in commodities, interdependence is modeled as the direct utility consequences of individual actions.

In the modification disequilibrium does not lead to infeasible outcomes, but to possibly unforeseen but nevertheless actual outcomes, to be corrected by individuals themselves rather than by 'market forces' in the guise of an auctioneer. This allows for a more strategic interpretation of price-taking behavior than the traditional interpretation of that term. Another significant feature of the modified description of interdependence is that externalities are a built-in feature of the environment. These changes stretch the concept of general equilibrium, inviting alternative approaches to address a more comprehensive collection of issues relating to the meaning of equilibrium, of competition, and its relation to economic efficiency, while retaining a link to their Walrasian origins.

8 Appendix

A restricted version of the result sufficient for Proposition 3, above, is:

Fenchel's Duality Theorem for Polyhedral Functions: (Rockafellar [1970], Theorem 31.1) Let $f:\mathbb{R}^m \to \mathbb{R}\cup\{-\infty\}$ be a proper polyhedral convex function and $g:\mathbb{R}^m \to \mathbb{R}\cup\{\infty\}$ a proper polyhedral concave function with (proper and polyhedral) conjugates denoted f^* and g^*. If either

$$\{x : f(x) < \infty\} \bigcap \{x : g(x) < \infty\} \neq \emptyset,$$

or

$$\{y : g^*(y) > -\infty\} \bigcap \{y : f^*(y) > -\infty\} \neq \emptyset,$$

there exists (x^0, y^0) such that

$$f(x^0) - g(x^0) = \inf_x \{f(x) - g(x)\} = \sup_y \{g^*(y) - f^*(y)\} = g^*(y^0) - f^*(y^0).$$

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